Test 1 Mathematical Structures TW1010
Wednesday October 3, 2018, 9:00-10:00

TU Delft

No calculators allowed. Write the solutions in the fields provided. The grade is (score+4)/4.

1a Write down the truth table for the expression $(p \vee q) \Rightarrow (p \wedge q)$. 3

Solution. The table is given by

p	q	$(p\vee q)$	$\Rightarrow$	$(p\wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

□

1b Is this expression a tautology? (Yes/No, because ...) 1

Solution. No, because in the column under the implication giving the truth value of the entire statement there are not only T's. □

2 Write down the negation of 4

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall y \in \mathbb{R} : |y-1| < \delta \Rightarrow |y^2 - 1| < \epsilon$$

in simplified form (the negation symbol itself is not allowed in your answer).
You only have to give your anwer, no explanation required.

Solution.

$$\exists \epsilon > 0 : \forall \delta > 0 : \exists y \in \mathbb{R} : |y-1| < \delta \wedge |y^2 - 1| \geq \epsilon.$$

□

3 Let A, B, and C be sets. Show that $(A \cap B) \setminus C = A \cap (B \setminus C)$. 9

Solution. Suppose $x \in (A \cap B) \setminus C$. Then $x \in (A \cap B)$ and $x \notin C$. As $x \in A \cap B$ we have $x \in A$ and $x \in B$. Since $x \in B$ and $x \notin C$ we have $x \in B \setminus C$. As $x \in A$ and $x \in B \setminus C$ we have $x \in A \cap (B \setminus C)$. Therefore $(A \cap B) \setminus C \subseteq A \cap (B \setminus C)$.

Now suppose $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Thus $x \in B$ and $x \notin C$. As $x \in A$ and $x \in B$ we have $x \in A \cap B$. As $x \in A \cap B$ and $x \notin C$ we conclude $x \in (A \cap B) \setminus C$. Thus $A \cap (B \setminus C) \subseteq (A \cap B) \setminus C$.

These two inclusions prove the equality of $(A \cap B) \setminus C = A \cap (B \setminus C)$. □

4 We define the relation on $\mathbb{N} \times \mathbb{N}$ as $(a, b)R(c, d)$ iff $ab - cd = 2k$ for some $k \in \mathbb{Z}$.

4a Show that R is an equivalence relation. 8

Solution. We need to show three properties

- Reflexivity. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be an arbitrary point. Then $ab - ab = 0 = 2 \cdot 0$ is of the desired form and thus $(a, b)R(a, b)$ holds.
- Symmetry. Suppose $(a, b)R(c, d)$ holds. Then $ab - cd = 2k$ for some $k \in \mathbb{Z}$. But then $cd - ab = -2k = 2 \cdot (-k)$ and as $-k \in \mathbb{Z}$ as well, we see that $(c, d)R(a, b)$.
- Transitivity. Suppose both $(a, b)R(c, d)$ and $(c, d)R(e, f)$ hold. Then $ab - cd = 2k$ and $cd - ef = 2l$ for some $k, l \in \mathbb{Z}$. But then $ab - ef = (ab - cd) + (cd - ef) = 2k + 2l = 2(k + l)$. As $k + l \in \mathbb{Z}$ it follows that $(a, b)R(e, f)$ holds.

As the relation is reflexive, symmetric and transitive we find that it is an equivalence relation. □

4b Determine the equivalence class $E_{(3,2017)}$ (that is, describe the elements of the class in as simple a way as possible). 2

Solution. The equivalence class of $(3, 2017)$ consists of all points which are in relation to $(3, 2017)$. Thus $(a, b) \in E_{(3,2017)}$ if $(a, b)R(3, 2017)$, that is, if $ab - 3 \cdot 2017 = 2k$ for some $k \in \mathbb{Z}$. This means that ab must be odd ($2k$ is even, and $3 \cdot 2017$ is odd). This happens if and only if both a and b are odd. Thus the equivalence class consists of all those points (a, b) with both a and b odd. □

5 Give the definition of the Cartesian product $A \times B$ of two sets A and B. 2

Solution. The Cartesian product $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. □

6 Give an example of a surjective function $f : \mathbb{R} \to [3, \infty)$. Be sure to show this is a good example, by showing that the values of the function are always in the codomain, and that the function is indeed surjective. 5

Solution. For example $f(x) = x^2 + 3$.

As all squares are positive we have $x^2 + 3 \geq 3$ and thus the function is well-defined (the results are all part of the co-domain $[3, \infty)$.

Moreover if $y \in [3, \infty)$, then take $x = \sqrt{y - 3}$ (note that $y - 3 \geq 0$). Then $f(\sqrt{y - 3}) = (\sqrt{y - 3})^2 + 3 = y - 3 + 3 = y$, and thus f is surjective. □

7 Write $\bigcup_{n=1}^{\infty} [1/n^2, 3 + 1/n]$ in the form of either a singleton set (a set containing one element), an interval, or a union of a finite number of singleton sets and intervals. 2
You only have to give the answer, no explanation required.

Solution. $(0, 4]$. □