Delft University of Technology
Faculty EEMCS
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Exam part 1 Real Analysis (TW2090)
6-11-2017; 13.30–15.30
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1. Let (M, d) and (N, ρ) be metric spaces and let $f : M \to N$ be a function.

(5) a. Complete the following definition: f is *continuous* in $x \in M$ if ....

(10) b. Assume that f is continuous in $x \in M$. Let (x_n) be a sequence in M with $x_n \xrightarrow{d} x$. Show that $f(x_n) \xrightarrow{\rho} f(x)$.

(5) 2. a. Let (M, d) be a metric space. Complete the following definition: a subset $A \subseteq M$ is called *totally bounded* if ...

(10) b. Give an example of a metric space (M, d) and a bounded subset $A \subseteq M$ which is not totally bounded. As always: prove all your assertions.

3. Let (M, d) be a metric space.

(5) a. Give the definition of the closure $\overline{A}$ of a set $A \subseteq M$.

(10) b. Using only the definition of the closure, prove the following equivalence for a set $A \subseteq M$: $\overline{A} = M \iff$ for all $x \in M$ and for all $\varepsilon > 0$ there exists a $y \in A$ such that $d(x, y) < \varepsilon$.

For a set $A \subseteq M$ and $\varepsilon > 0$ we define

$$A(\varepsilon) = \{x \in M : \exists y \in A \text{ such that } d(x, y) < \varepsilon\}.$$

(5) c. Show that $A(\varepsilon)$ is open.

Let (A_n) be a sequence of subsets of M such that for all $n \geq 1$ one has $A_n \subseteq A_{n+1}$ and $\overline{\bigcup_{n\geq 1} A_n} = M$.

(6) d. Use (b) to show that for each $\varepsilon > 0$ one has $M = \bigcup_{n\geq 1} A_n(\varepsilon)$.

(6) e. From now on assume that M is compact. Show that for every $\varepsilon > 0$ there exists an $n \geq 1$ such that $M = A_n(\varepsilon)$.

4. Let X be a nonempty set and let $B(X)$ be the vector space of bounded functions $f : X \to \mathbb{R}$. On $B(X)$ we define $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

(6) a. Show that $\|\cdot\|_\infty$ is a norm on $B(X)$.

(12) b. Prove that $(B(X), \|\cdot\|_\infty)$ is complete.

(10) 5. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that for every integer $n \geq 0$,

$$\int_0^1 f(x)x^n dx = 0.$$

Show that $f = 0$.
Hint: First explain why for all polynomials p one has $\int_0^1 f(x)p(x)dx = 0$ and then use Weierstrass' theorem to find that $\int_0^1 (f(x))^2 dx = 0$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END