Midterm Exam OPTIMIZATION (TW2020)

5 October 2018

13:30 – 15:30

This midterm exam consists of 4 questions on 2 pages. You can earn 45 points in total. Your grade is determined by adding 5 to the total number of obtained points and to subsequently divide by 5. You only earn points when you use the **right method** and describe all **steps and arguments** clearly. You may only use a non-graphical calculator. Success!

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1. (10 points) In the afternoon of the 5th of October, E exams need to take place at the TU Delft. There are R rooms available, each with a limited capacity. Let c_r be the capacity of room r (the number of students that can make their exam there) and let s_e be the number of students that have signed up for exam e. It is possible to have different exams in the same room. However, it is not possible to split up an exam over multiple rooms. In order to save costs, the university wants to minimize the number of rooms used. Give an **ILP formulation** of this problem.

2. Consider the following LP. Answer the questions below **without** applying the Simplex method.

$$
\begin{array}{lll}
\min & 3x_1 + 4x_2 + 4x_3 + 4x_4 & \\
s.t. & 2x_1 + x_2 + 3x_3 + x_4 \geq 5 & \\
 & x_1 + x_2 + x_3 + 2x_4 = 6 & \\
 & x_1, x_2, x_3, x_4 \geq 0 &
\end{array}
$$

   (a) (3 points) Prove that 11 is a lower bound on the objective function value of an optimal solution.

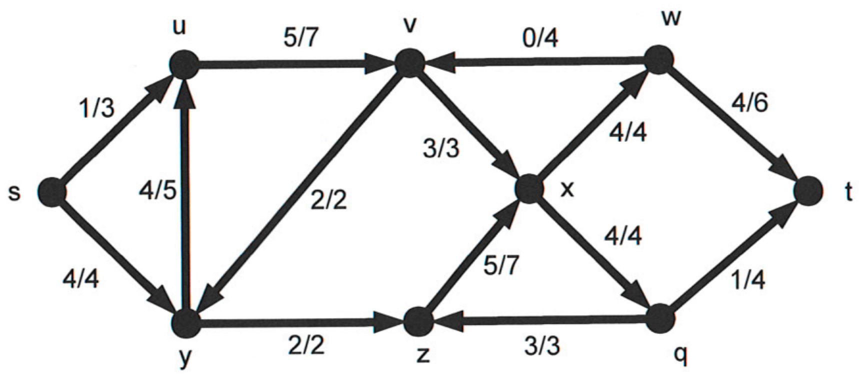
   (b) (3 points) Put the LP in standard form and find the basic solution with x_1 and x_4 as basic variables.

   (c) (3 points) Prove that this LP has an optimal solution.

   (d) (3 points) Write down the dual (D) of this LP (before it was put in standard form).

   (e) (3 points) Prove that (D) has an optimal solution, by directly applying a theorem, and give a lower and upper bound on the optimal value of the objective function of (D).

3. Consider thet network below. Each arc a has label f_a/b_a with f_a the current flow on arc a and b_a the capacity of arc a.



(a) (4 points) Use **complementary slackness** to show that the current flow is optimal.

(b) (6 points) Suppose that the capacity b_{yz} of arc (y, z) is increased to 3. Find a larger flow by applying one iteration of the **Ford-Fulkerson** algorithm. Is the obtained flow **optimal?**

4. Let A be an $m \times n$ matrix, $\boldsymbol{b} \in \mathbb{R}^m$ and

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}.$$

(a) (4 points) **Prove** that P is a convex set (using the definition of convex set).

(b) (6 points) **Prove** that $P = \emptyset$ if and only if there exists a $\boldsymbol{y} \in \mathbb{R}^m$ with

$$\begin{aligned} \boldsymbol{y}^T\boldsymbol{b} &< 0 \\ \boldsymbol{y}^T A &\geq \boldsymbol{0}^T \\ \boldsymbol{y} &\geq \boldsymbol{0}. \end{aligned}$$