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Exam part 2 Real Analysis (TW2090)
23-1-2018; 9.00–11.00
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1. Let S be a set.

(4) a. Complete the following definition: a set $\mathcal{A} \subset \mathcal{P}(S)$ is called a σ-algebra if ....

Let I be an index set and assume that for each $i \in I$, $\mathcal{A}_i$ is a σ-algebra on S.

(6) b. Show that $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra.

2. Let λ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $f : \mathbb{R} \to [0, \infty)$ be given by $f(x) = e^{-x^2} \sin^2(x)$.

(7) a. Prove that f is integrable

Define $\nu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\nu(B) = \int_B f \, d\lambda$.

(8) b. Show that ν is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(10) c. Let $g : \mathbb{R} \to [0, \infty)$ be a measurable function. Show that $\int_{\mathbb{R}} g \, d\nu = \int_{\mathbb{R}} gf \, d\lambda$.
Hint: First consider a simple function $g : \mathbb{R} \to [0, \infty)$.

3. Let $(S, \mathcal{A}, \mu)$ be a measure space.

(6) a. Let $(A_n)_{n\geq 1}$ be a sequence of sets in $\mathcal{A}$. Show that $\mu\Big(\bigcup_{n\geq 1} A_n\Big) \leq \sum_{n\geq 1} \mu(A_n)$.

For each $n \geq 1$ let $f_n : S \to \mathbb{R}$ be a measurable function. Assume that there exists an $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f_n \to 0$ pointwise on N^c. For each $n, j \geq 1$ define $A_{n,j} = \bigcup_{m\geq n} \{s \in S : |f_m(s)| \geq 1/j\}$.

(6) b. Prove that for each $j \geq 1$ one has $\mu\big(\bigcap_{n\geq 1} A_{n,j}\big) = 0$.

From now on assume $\mu(S) < \infty$ and $\varepsilon > 0$.

(6) c. Show that for each $j \geq 1$ there exists an n_j such that $\mu(A_{n_j,j}) \leq \frac{\varepsilon}{2^j}$
Hint: Use a convenient theorem for a decreasing sequence of sets from the lecture notes.

Let $B := \bigcup_{j\geq 1} A_{n_j,j}$ where n_j is as in (c).

(6) d. Show that $\mu(B) \leq \varepsilon$ and explain why $f_n \to 0$ uniformly on B^c.

(20) 4. State and prove the dominated convergence theorem.

5. Let $f : [0, 2\pi) \to \mathbb{R}$ be defined by $f(x) = \mathbf{1}_{[\pi,2\pi)}(x)$.

(3) a. Calculate the $L^2(0, 2\pi)$-norm of f.

(4) b. Show that $s_n(f) : [0, 2\pi] \to \mathbb{C}$ (the n-th partial sum of the Fourier series of f) is given by $s_n(f) = \sum_{|k|\leq n} c_k e_k$ with $c_0 = 1/2$, $c_k = 0$ if $k \neq 0$ is even, and $c_k = \frac{i}{\pi k}$ if k is odd.

(4) c. Calculate $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$ using (a), (b) and Parseval's identity.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END

2. a. Note that f is continuous and hence measurable. To prove that f is integrable by a theorem from the lecture notes we note that

$$\int_{\mathbb{R}} |f| \, \mathrm{d}\lambda = \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x$$

Since $|f(x)| \leq e^{-x^2} \leq \max\{1, e^{-|x|}\}$, by the properties of the Riemann integral we can estimate

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x &= \lim_{t\to-\infty} \int_t^{-1} f(x) \, \mathrm{d}x + \int_{-1}^{1} f(x) \, \mathrm{d}x + \lim_{t\to\infty} \int_1^t f(x) \, \mathrm{d}x \\ &\leq \lim_{t\to-\infty} \int_t^{-1} e^x \, \mathrm{d}x + \int_{-1}^{1} 1 \, \mathrm{d}x + \lim_{t\to\infty} \int_1^t e^{-x} \, \mathrm{d}x = 2e^{-1} + 2 < \infty.\end{aligned}$$

3. a. See lecture notes

b. We claim that $\bigcap_{n\geq 1} A_{n,j} \subseteq N$. Indeed, if $s \in \bigcap_{n\geq 1} A_{n,j}$, then for all $n \geq 1$ we have $s \in A_{n,j}$. From the definition of $A_{n,j}$ we obtain that for all $n \geq 1$ there exists an $m \geq n$ such that $|f_m(s)| \geq 1/j$. Therefore, $f_n(s) \nrightarrow 0$ and we can conclude that $s \in N$. From the claim we see that $\mu\Big(\bigcap_{n\geq 1} A_{n,j}\Big) \leq \mu(N) = 0$.

c. Fix $j \geq 1$. From the definition it follows that $(A_{n,j})_{n\geq 1}$ is decreasing. Therefore, $A_{n,j} \downarrow \bigcap_{n\geq 1} A_{n,j}$. Since $\mu(A_{1,j}) \leq \mu(S) < \infty$, we can apply a theorem from the lecture notes to obtain $\mu(A_{n,j}) \to \mu\Big(\bigcap_{n\geq 1} A_{n,j}\Big) = 0$. Therefore, we can find n_j such that $\mu(A_{n,j}) \leq \frac{\varepsilon}{2^j}$.

d. Let n_j be as in (c). By the σ-additivity of μ (see (a)), we obtain

$$\mu(B) \leq \sum_{j\geq 1} \mu(A_{n_j,j}) \leq \sum_{j\geq 1} \frac{\varepsilon}{2^j} = \varepsilon.$$

Now if $s \in B^c = \bigcap_{j\geq 1} A_{n_j,j}$, then for all $j \geq 1$, $s \in A_{n_j,j}^c$. Therefore, for all $j \geq 1$ for all $m \geq n_j$, $|f_m(s)| < \frac{1}{j}$. Now if $\delta > 0$, then choosing $j \geq 1$ such that $1/j \leq \delta$, we find that for all $m \geq n_j$, for all $s \in B^c$, $|f_m(s)| \leq 1/j < \delta$.

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