Exam OPTIMIZATION (TW2020)

30 October 2018

13:30 – 16:30

This exam consists of 5 questions on 2 pages. You can earn 70 points in total. Your grade is determined by dividing the total number of obtained points by 7. You only earn points when you use the **right method** and describe all **steps and arguments** clearly. You may only use a non-graphical calculator. Success!

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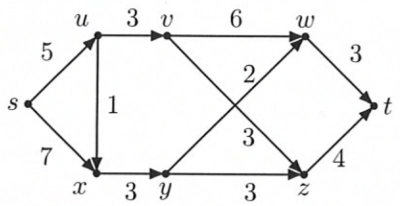
1. Consider the following directed graph in which the label of each arc a indicates its length.



   (a) (5 points) Find the **length** of a shortest s-t path using the algorithm of **Dijkstra**.

   (b) (5 points) Use **Complementary Slackness** to **find** a shortest s-t path and to **prove** that no shorter s-t path exists.

2. (10 points) Consider an undirected graph $G = (V, E)$ with a length function $\ell : E \to \mathbb{R}$ on its edges such that $\ell(e) \neq \ell(f)$ for all $e, f \in E$ with $e \neq f$. Prove that there exists **at most one** minimum spanning tree of G with respect to ℓ.

3. A computational biologist wants to study a large data set of protein sequences. However, the data set is too large to study every single one of the proteins. Luckily, many protein sequences are similar. Therefore, the computational biologist wants to find a smallest possible "representative set" of proteins. For each protein, such a representative set should either contain that protein or all proteins that are similar to it (or both). If a protein is not similar to any other protein then it should also be included in the set.

   To find such a representative set, he/she first checks for each pair of proteins whether they are similar or not and makes the following data. Let P be the set of proteins. *Similar protein lists* for P contain, for each $p \in P$, a list $S_p \subseteq P \setminus \{p\}$ of proteins that are similar to p, such that for each pair p, q of proteins, $q \in S_p$ if and only if $p \in S_q$.

   After that, he/she wants to find a *representative set* for P, which is a subset $R \subseteq P$ such that (i) $S_p \subseteq R$ for each protein $p \in P \setminus R$ and (ii) $p \in R$ for each protein $p \in P$ with $S_p = \emptyset$. Therefore, the following problem needs to be solved.

   REPRESENTATIVE PROTEINS
   Given: a set P of proteins and similar protein lists S_p for $p \in P$.
   Find: a smallest possible representative set R for P.

(a) (10 points) Give an **ILP formulation** of REPRESENTATIVE PROTEINS.

(b) (10 points) Now consider the decision variant of this problem, which is defined as follows.

REPRESENTATIVE PROTEINS DECISION
Given: a set P of proteins, similar protein lists S_p for $p \in P$ and an integer k.
Decide: does there exist a representative set R for P of size at most k?

Prove that REPRESENTATIVE PROTEINS DECISION is **NP-complete**, using that the problem VERTEX COVER DECISION below is NP-complete.

VERTEX COVER DECISION
Given: a graph $G = (V, E)$ and an integer k'.
Decide: does there exist a subset $U \subseteq V$ containing at most k' vertices, such that each edge from E is incident to at least one vertex from U?

4. Consider the LP below.

$$\begin{array}{llcrcl} \min & 2x_1 & + & 7x_2 & & \\ \text{s.t.} & x_1 & + & 3x_2 & = & 12 \\ & & & 2x_2 & \geq & 3 \\ & x_1 & - & x_2 & \leq & 7 \\ & x_1, x_2 \geq 0 & & & & \end{array}$$

(a) (10 points) Solve this LP using the **two phase Simplex** method. (Hint: if there are multiple options for the entering basic variable, choose the first possible entering basic variable from the following list: $(s_3, x_2, x_1, x_1^a, s_2, x_2^a)$.)

(b) (5 points) Formulate the **dual**. You do not need to motivate your answer.

(c) (5 points) Find an optimal solution of the dual using the **complementary slackness** conditions and the optimal solution found in part (a).

5. Consider the ILP below.

$$\begin{array}{lllcrcl} \max & z = & x_1 & + & 8x_2 & & \\ \text{s.t.} & & 3x_1 & - & 4x_2 & \leq & 0 \\ & & 5x_1 & - & x_2 & \leq & 10 \\ & & & & x_2 & \leq & 4 \\ & & x_1, x_2 \geq 0 & & & & \\ & & x_1, x_2 \in \mathbb{Z} & & & & \end{array}$$

The optimal Simplex tableau of the LP relaxation is as follows:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
s_1	$38/5$	0	0	1	$-3/5$	$17/5$
x_1	$14/5$	1	0	0	$1/5$	$1/5$
x_2	4	0	1	0	0	1
$-z$	$-174/5$	0	0	0	$-1/5$	$-41/5$

(a) (5 points) Find the **Gomory cutting plane** corresponding to the row of s_1 and express it in the original variables x_1, x_2.

(b) (5 points) Find an optimal solution of the LP-relaxation of the above ILP with the Gomory cutting plane found in part (a) added, using the **dual Simplex method**.

End exam