**TU Delft/Delft Insititute Applied Mathematics/Applied Probability**

**Exam Advanced Probability TW 3560**
**10th April 2019, 13:30-16:30**
Lecturer: Dr. Wioletta Ruszel
Second reader: Dr. Alessandra Cipriani

- The exam is a closed book exam. You may use a simple non-graphical calculator.
- All solutions should be well-documented and explained, otherwise full points will not be awarded.
- In the **first part** there are 5 questions. Every correct answer gives 1 point, you can reach maximally 5 points.
- The **second part** will be considered if in the first part the student scored more than 3 points. For the second part the points are distributed as follows:

| | Exercise 1 | Exercise 2 | Exercise 3 | Exercise 4 |
|---|---|---|---|---|
| Points | 1-1-1- 5 | 8 | 1-1-2-1-2 | 2-5 |

- The total number of points is 35. The grade is calculated in the following way:

$$\text{grade} = \min\left(\frac{1}{3.5}(\sharp\text{points}) + \text{bonus point}, 10\right).$$

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**Part I** (5 P.)
Indicate which of the following statements are true and explain why. In case, compute the quantity which is asked.

1. Let $\{A_n; n \geq 1\}$ be a sequence of events. Then $\limsup_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$. F.
2. If $X$ is a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$, then for all $A \in \mathscr{F}$, we have that $X^{-1}(A) \in \mathscr{B}$, where $\mathscr{B}$ is the Borel sigma algebra.
3. Let $X_0, ..., X_n, ..$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_0) = 0$ and $\mathbb{E}(X_0^2) = 2$, then by the weak law of large numbers:
$$\frac{\sum_{i=0}^{2^n-1} X_i}{2^n} \underset{n\to\infty}{\longrightarrow} 0$$
in probability.
4. If a sequence of i.i.d. random variables $(X_n)_{n\geq 1}$ with mean 0 satisfies $\frac{1}{s_n^4}\sum_{k=1}^{n} \mathbb{E}(X_k^4) \underset{n\to\infty}{\longrightarrow} 0$ where $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$ and $\sigma_k^2 = \mathbb{E}(X_k^2)$ then $\frac{\sum_{k=1}^{n} X_k}{s_n} \xrightarrow{d} N(0,1)$ as $n \to \infty$.
5. Let $f(x) = 2^{-x}$ and $X \sim Geo(0.5)$ is a geometric random variable, compute $\mathbb{E}(f(X))$.

## Part II

### Exercise 1 (8 P.)

1. Let $X$ be a random variable with $\mathbb{P}(X > 0) > 0$. We will prove that there exists $\delta > 0$ such that $\mathbb{P}(X > \delta) > 0$ by showing (a)-(c).

   (a) Let $A = \{X > 0\}$. Find a sequence of events $A_n$ such that $\bigcup_{n=1}^{\infty} A_n = A$.

   (b) Show that there exists $\epsilon > 0$ such that $\lim_{n\to\infty} \mathbb{P}(A_n) = \epsilon$.

   (c) Find $\delta = \delta(n)$ such that $\mathbb{P}(X > \delta) > 0$.

2. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, all defined on the same probability space such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = 1$ for all $i \in \mathbb{N}$. Prove that

$$\mathbb{P}(X_n > n \text{ i.o. }) = 0.$$

### Exercise 2 (8 P.)

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, all defined on the same probability space, such that for all $n \in \mathbb{N}$: $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$. Show that $X_n \to 0$ almost surely as $n \to \infty$ if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

### Exercise 3 (7 P.)

Adam has $N$ batteries available for an electrical device. Once the battery is empty he replaces it with another one. The lifetime $L_i$ of one battery $i$ is independent of the lifetime $L_j$ of battery $j$ $(i \neq j)$ and is exponentially distributed with mean 2 (rescaled out 100 hours, which means that the actual mean lifetime is 200 hours).

1. What is the probability density function of $L_1 + L_2$?

2. Let $N$ be a random variable with sample space $\{1, 2, 3\}$ and distribution

$$\mathbb{P}(N = 2) = \mathbb{P}(N = 3) = \frac{1}{4}, \quad \mathbb{P}(N = 1) = \frac{1}{2}$$

   independent of $L_1, L_2, L_3$. What is $\mathscr{L}(X_N)$, the law of $X_N = \sum_{i=1}^{N} L_i$?

3. What is the expected value of $X_N$?

4. Give a non-trivial upper bound on the probability that the device works at least 1000 hours. → Chebyshev?

5. Berta has an analogous device and 2 batteries at home. What is the probability that Berta's device will work longer Adam's device with $N$ batteries, distributed as in point (2)?

### Exercise 4 (7 P.)

1. State the Theorem of Lindeberg-Lévy-Feller.

2. Let $X_1, X_2, ..$ be a sequence of independent random variables such that $X_n \sim N(0, \sigma_n^2)$ for all $n \in \mathbb{N}$. Define $\sigma_n^2 = e^{-n}$ and show that the Lindeberberg conditions do not hold. Show that nevertheless, by appropriate rescaling, $S_n = X_1 + \ldots + X_n$ converges to a standard normal variable. Does it contradict the theorem of Lindeberg-Lévy-Feller?

↳ finite variances?