Delft University of Technology
Faculty EEMCS
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Exam part 2 Real Analysis (TW2090)
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1. Let S be a set.

(4) a. Complete the following definition: A family $\mathcal{R}$ of subsets of S is called a *ring* if ....

(4) b. Complete the following definition: A mapping $\mu : \mathcal{R} \to [0, \infty]$ is called *additive* if ....

From now on we assume that $\mathcal{R}$ is a ring on S and $\mu : \mathcal{R} \to [0, \infty]$ is additive.

(3) c. Prove that for $A, B \in \mathcal{R}$ with $\mu(A) < \infty$ and $A \subseteq B$ one has $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(5) d. Prove that for $A, B \in \mathcal{R}$ with $\mu(A) < \infty$ one has $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

2. Consider the following collection $\mathcal{B}_0 = \{(x, \infty) : x \in \mathbb{R}\})$ of subsets of $\mathbb{R}$.

(6) a. Show that $\sigma(\mathcal{B}_0)$ contains all open intervals.

(4) b. Use Lemma I (see next page) to show that every open set $O \subset \mathbb{R}$ is the union of countably many open intervals.

(4) c. Show that $\sigma(\mathcal{B}_0) = \mathcal{B}(\mathbb{R})$.

3. Let $(S, \mathcal{A}, \mu)$ be a measure space.

(4) a. Complete the following definition: a function $f : S \to \mathbb{R}$ is called *measurable* if ....

(4) b. Assume that for all $r \in \mathbb{R}$, $f^{-1}\big((r, \infty)\big) \in \mathcal{A}$. Use Lemma II (see next page) and 2c to deduce that f is measurable.

Let I be an index set (not necessarily countable!) and let for each $i \in I$, $f_i : \mathbb{R} \to [0, 2019]$ be continuous.

(6) c. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sup_{i \in I} f_i(x)$. Use (b) to prove that f is measurable.

4. Let λ be the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Let $f : \mathbb{R}^d \to [0, \infty]$ be a measurable function. For $h \in \mathbb{R}^d$ define the translation $f_h : \mathbb{R}^d \to [0, \infty]$ by $f_h(x) = f(x - h)$.

(4) a. Explain why f_h is measurable.

(10) b. Show that $\displaystyle\int_{\mathbb{R}^d} f_h \, d\lambda = \int_{\mathbb{R}^d} f \, d\lambda$.

(4) 5. a. State the monotone convergence theorem.

(11) b. State and prove Fatou's lemma.

(10) c. On $\mathbb{R}$ consider the Lebesgue measure λ. Assume $f : \mathbb{R} \to \mathbb{R}$ is integrable. Use the dominated convergence theorem to show that $\lim_{n\to\infty} \|f \mathbf{1}_{[n,n+1]}\|_1 = 0$.

(7) 6. Let $f, g : [0, 2\pi] \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{2}x^2 - \pi x + \frac{1}{3}\pi^2 \quad \text{and} \quad g = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{n^2} e_n.$$

With the help of the integral identities (see next page) show that $f = g$ in $L^2(0, 2\pi)$.

See also the next page.

Lemma I (Lindelöf) *Let* $A \subseteq \mathbb{R}^d$. *For each* $i \in I$ *let* $O_i \subseteq \mathbb{R}^d$ *be open. If* $A \subseteq \bigcup_{i \in I} O_i$, *then there exists a countable* $J \subseteq I$ *such that* $A \subseteq \bigcup_{i \in J} O_i$

Lemma II *Let* $(S, \mathcal{A})$ *and* $(T, \mathcal{B})$ *be two measurable spaces and let* $f : S \to T$. *Suppose* $\mathcal{F} \subseteq \mathcal{B}$ *is such that* $\sigma(\mathcal{F}) = \mathcal{B}$. *If* $f^{-1}(F) \in \mathcal{A}$ *for all* $F \in \mathcal{F}$, *then* f *is measurable.*

Integral identities

$$\int_0^{2\pi} x e^{-inx} dx = \frac{2\pi i}{n}, \quad n \in \mathbb{Z} \setminus \{0\},$$
$$\int_0^{2\pi} x^2 e^{-inx} dx = \frac{4\pi^2 i}{n} + \frac{4\pi}{n^2}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END