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Time Series and Extreme Value theory WI4230
Final Exam - April 2018

Please provide your student card on the table, ready for inspection.
Mobiles, tablets and similar objects must be switched off. The exam is invalidated if you cheat.
Please write with a pen. Please write your name, surname and student number on all your papers.

The exam is on the extreme value theory part of the course. You have in total **3 hours for 6 questions**.

In the whole exam, the tail quantile function is denoted by $U(t)$, $t > 1$. And, $X_{i,n}$ denotes the i-th order statistics of a random sample with size n for $i = 1, \ldots, n$

1. $U(t) = \sqrt{t-1}$, $t \geq 1$. $\gamma = 1/2$. $a_n = \frac{\sqrt{n}}{2}$ and $b_n = \sqrt{n-1}$ (these two are not unique). $G(x) = \exp(-(1+x/2)^{-2})$.

2.

$$\frac{1}{k}\sum_{i=1}^{k}\frac{X_{n-k,n}}{X_{n-i+1,n}} \stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\frac{U(Y_{n-k,n})}{U(Y_{n-i+1,n})} \approx \frac{1}{k}\sum_{i=1}^{k}\left(\frac{Y_{n-k,n}}{Y_{n-i+1,n}}\right)^{\gamma} = \frac{1}{k}\sum_{i=1}^{k}\exp\left(\gamma(\log Y_{n-k,n} - \log Y_{n-i+1,n})\right)$$
$$\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\exp\left(-\gamma(E_{n-i+1,n} - E_{n-k,n})\right) \stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\exp\left(-\gamma E^*_{k-i+1,k}\right)$$
$$= \frac{1}{k}\sum_{i=1}^{k}\exp\left(-\gamma E^*_i\right) \stackrel{p}{\to} E(e^{-\gamma E_1}) = \int_0^{\infty} e^{-\gamma x}e^{-x}dx = \frac{1}{1+\gamma}.$$

3.
- True. the necessary and sufficient condition for $F \in D(G_\gamma)$ implies that there is only one γ.
- False. By choosing different a_n and b_n, the same F leads to different G. G is unique up to a location-scale transformation. The shape parameter, that is γ, is unique. However, the location and scale parameter depends on the choice of a_n and b_n.
- True. Many ways to show this. If $F \in D(G)$ then $H(x) = F(c_1x + c_2)$ for any $c_1 > 0$ and $c_2 \in R$ will also be in the same max domain of attraction.
- False. The three k's in $X_{n-k,n}$, $\hat{a}^M(n/k)$ and $\frac{k}{np}$ have to be the same. The k in $\hat{\gamma}^M(k)$ can be chosen different. This can be seen from the approximation of tail function.

  The moment estimator of a tail quantile $U(1/p)$ is given by:

$$\hat{U}^M(\frac{1}{p}) = X_{n-k,n} + \hat{a}^M(n/k)\frac{(\frac{k}{np})^{\hat{\gamma}^M(k)} - 1}{\hat{\gamma}^M(k)}.$$

  In the constructing blocks of this estimator, there are four parts depending on k: $X_{n-k,n}$, $\hat{a}^M(n/k)$, $\frac{k}{np}$ and $\hat{\gamma}^M(k)$.

4. There are different solutions to this problem. First, one can use MLE to estimate γ, this method does not require the tail observations to be positive since it only depends on the exceedens $X_{n-i+1,n} - X_{n-k,n}$,

which are always positive.
Or, one can shift the data up such that the tail observations are positive and then apply Hill or moment estimator. The reason is that if $X \in D(G_\gamma)$ then $X + c \in D(G_\gamma)$ for any constant c.

For estimating the tail quantile $U(10^5)$, the estimator of $\hat{U}(1/p) = X_{n-k,n}(k/np)^{\hat{\gamma}}$ does not work on the original data. Because likely $X_{n-k,n}$ is negative, and then your estimate $\hat{U}(1/p) << \hat{U}(n/k)$ which is not logical. But you can shift the data to estimate $U_Y(10^5)$ where $Y = X + 10$. And shift the estimator back.

5. Let $\{(X_i, Y_i), i = 1, \ldots, n\}$ be n i.i.d. bivariate random vectors. Suppose that the distribution of (X_1, Y_1) is in a bivariate max domain of attraction, that is, there exist two sequences of positive numbers $a_{1,n}$ and $a_{2,n}$, two sequences of real numbers $b_{1,n}$ and $b_{2,n}$, and a distribution function G with non-degenerate marginals, such that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{1\le i\le n} X_i - b_{1,n}}{a_{1,n}} \le x, \frac{\max_{1\le i\le n} Y_i - b_{2,n}}{a_{2,n}} \le y\right) = G(x,y), \tag{1}$$

for all continuity points (x, y) of G.

(a) Let F_1 and U_1 be the cdf and tail quantile function of X. From (1), $\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{1\le i\le n} X_i - b_{1,n}}{a_{1,n}} \le x\right) = G(x, \infty)$. So $F_1 \in D(G(x, \infty))$. Thus there exists a positive function $a(t)$ such that $\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}$. One can choose, $a_n = a(n)$ and $b_n = U(t)$ to get the intended marginal distribution for the limit.

(b) most of you get this right. Similar to the proof for Corollary for Theorem 5.

6. Ψ is a spectral measure because it is finite and it satisfies the side condition. The bivariate extreme value distribution is

$$G(x, y) = \exp\left(-\frac{1}{x} - \frac{1}{y} + \frac{1}{\sqrt{x^2 + y^2}}\right).$$