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Time Series and Extreme Value theory WI4230
Final Exam - April 2018

Please provide your student card on the table, ready for inspection.
Mobiles, tablets and similar objects must be switched off. The exam is invalidated if you cheat.
Please write with a pen. Please write your name, surname and student number on all your papers.

The exam is on the extreme value theory part of the course. You have in total **3 hours for 6 questions**.

In the whole exam, the tail quantile function is denoted by $U(t)$, $t > 1$. And, $X_{i,n}$ denotes the i-th order statistics of a random sample with size n for $i = 1, \ldots, n$

1. (10 points) Show that $F(x) = 1 - \frac{1}{(1+x^2)}$, $x \geq 0$ is in a max domain of attraction. What is the extreme value index of this distribution? Find $a_n > 0$ and $b_n \in \mathbf{R}$ and a non-degenerate distribution function G such that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{i=1,\ldots,n} X_i - b_n}{a_n} \leq x\right) = G(x),$$

at the continuity points of G, where $\{X_i, i = 1, \ldots, n\}$ are i.i.d. random variables from F.

2. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution function $F \in D(G_\gamma)$, where $\gamma > 0$. Apart from the Hill method, one can also use the following estimator for γ:

$$\hat{\gamma} := \left(\frac{1}{k}\sum_{i=1}^{k} \frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^{-1} - 1,$$

where $k = k(n)$ is a sequence of integers such that $k \to \infty$ and $k/n \to 0$ as $n \to \infty$. Show that

$$\hat{\gamma} \xrightarrow{P} \gamma,$$

as $n \to \infty$.
Hint: You might use that $\{E_{n-i,k} - E_{n-k,n}\}_{i=0}^{k-1} =^d \{E^*_{k-i,k}\}_{i=0}^{k-1}$, where $\{E^*_1, \ldots, E^*_k\}$ is a random sample from $Exp(1)$.

3. (10 points) Determine the correctness of each statement and explain why.

- For any given F, if $F \in D(G_\gamma)$, then γ is unique.
- For any given F, if $F \in D(G_\gamma)$, then G is unique.
- For any given extreme value distribution G, there are infinite many distributions F's such that $F \in D(G)$.
- The moment estimator of a tail quantile $U(1/p)$ is given by:

$$\hat{U}^M(\frac{1}{p}) = X_{n-k,n} + \hat{a}^M(n/k)\frac{(\frac{k}{np})^{\hat{\gamma}^M(k)} - 1}{\hat{\gamma}^M(k)}.$$

  In the constructing blocks of this estimator, there are four parts depending on k: $X_{n-k,n}$, $\hat{a}^M(n/k)$, $\frac{k}{np}$ and $\hat{\gamma}^M(k)$. All these k's are necessarily the same.

4. (10 points) Based on a data set with 5000 observations, the aim is to estimate the extreme value index and the tail quantile $U(10^5)$. The prior knowledge indicates that the data has a heavy right tail, that is $X \in D(G_\gamma)$ and $\gamma > 0$. As shown in Figure 1, there are only a few observations that are positive, which means that the Hill estimator can not be directly applied because it uses logarithm of the observations. What is your solution for estimating γ and $U(10^5)$? Describe your procedure as well as the reasoning behind it.

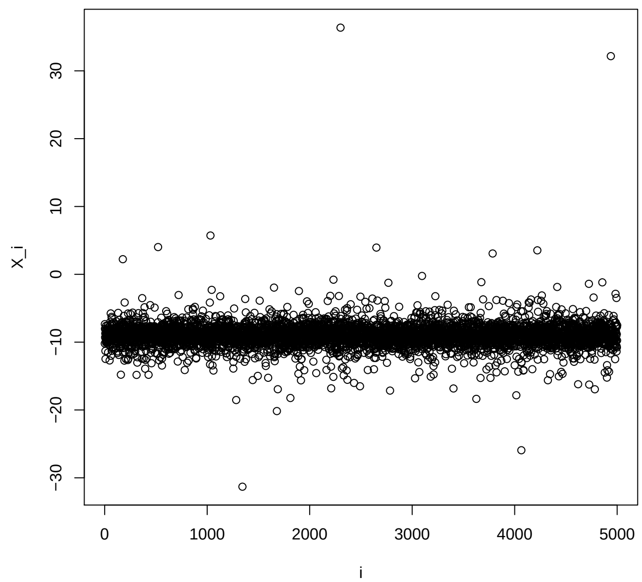

Figure 1: 5000 observations from a heavy tailed distribution

5. (10 points) Let $\{(X_i, Y_i), i = 1, \ldots, n\}$ be n i.i.d. bivariate random vectors. Suppose that the distribution of (X_1, Y_1) is in a bivariate max domain of attraction, that is, there exist two sequences of positive numbers $a_{1,n}$ and $a_{2,n}$, two sequences of real numbers $b_{1,n}$ and $b_{2,n}$, and a distribution function G with non-degenerate marginals, such that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{1\le i\le n} X_i - b_{1,n}}{a_{1,n}} \le x, \frac{\max_{1\le i\le n} Y_i - b_{2,n}}{a_{2,n}} \le y\right) = G(x, y), \tag{1}$$

for all continuity points (x, y) of G.

(a) How to choose the normalising sequences $a_{i,n}$ and $b_{i,n}$, $i = 1, 2$ such that (1) holds with $G(x, \infty) = \exp(-(1 + \gamma_1 x)^{-1/\gamma_1})$ and $G(\infty, y) = \exp(-(1 + \gamma_2 y)^{-1/\gamma_2})$?

(b) Prove that if the normalising sequences are chosen as in Question (a), then for any $x > 0$ and $y > 0$,

$$\lim_{n\to\infty} n\mathbf{P}\left(\left(1 + \gamma_1 \frac{X_1 - b_{1,n}}{a_{1,n}}\right)^{1/\gamma_1} > x \quad \text{or} \quad \left(1 + \gamma_1 \frac{Y - b_{2,n}}{a_{2,n}}\right)^{1/\gamma_2} > y\right) = -\log G\left(\frac{x^{\gamma_1} - 1}{\gamma_1}, \frac{y^{\gamma_2} - 1}{\gamma_2}\right)$$

You are not supposed to apply the result of Theorem 8 to solve this question.

6. (10 points) Let Ψ be a measure on $[0, \pi/2]$ such that $\Psi([0, \theta]) = \frac{3}{2}\sin^2\theta$. Is Ψ a spectral measure that gives rise to a bivariate extreme value distribution? If yes, what is the corresponding bivariate extreme value distribution? If not, can you revise it in a way that it satisfies the conditions of a spectral measure?