Contents

1 The Heat Equation

Let's consider a rod. Let $e(x,t)$ be the thermal energy at a point x at time t, A the surface of the cross-section and Δx be a small length. Then the heat energy between x and $x+\Delta x$ at time t is approximately $e(x,t)A\Delta x$. The change of this energy is determined by the flux $\phi(x,t)$ per surface and the sum of sources in that region $Q(x,t)$ in the following way:

$$\partial_t e(x,t)A\Delta x = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)A\Delta x.$$

This implies

$$\partial_t e(x,t) = -\partial_x \phi(x,t) + Q(x,t)$$

by letting $\Delta x \to 0$. Now let $u(x,t)$ denote the actual temperature at x at time t. Then u is related to e via

$$e(x,t) = \rho(x)c(x)u(x,t),$$

where c is the specific heat and ρ the density. Also, the heat flux is related to the temperature and is given by

$$\phi(x,t) = -K_0(x)\partial_x u(x,t),$$

where K_0 is the thermal conductivity of the material, which is called Fourier's law of heat conduction. Now, if we substitue this in our equation we get

$$\rho c \partial_t (u) = \partial_{xx} (K_0 u) + Q.$$

If we assume $Q = 0$ and the material parameters to be constant along the rod we obtain $u_t = ku_{xx}$ with the thermal diffusivity $k = \frac{K_0}{c\rho}$.

It is still unclear how u behaves at the boundary. There are three cases

- If we prescribe a temperature at the boundary we get the condition $u = \alpha$ on the boundary.
- If the boundary is insulated we get the condition $u_x = 0$ on the boundary.
- If the rod is in contact with another material at the boundary there is a heat flux which tries to equalize the temperature difference. So at 0 one gets for example $\phi = -H(u - u_B)$ with $H > 0$. But this means $K_0 u_x = H(u - u_B)$ at 0. Note that the sign switches at the other end.

In higher dimensions the heat equation takes the form $u_t = k\Delta u$.

2 Laplace Equation

The Laplace equation describes the steady state of the heat or wave equation, namely $\Delta u = 0$. Functions satisfying the Laplace equation have the so called mean value property

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, dS(y).$$

This implies the maximum principle, which states that u attains its maximum on the boundary of the domain. By this we can prove for example uniqueness of solutions. If one imposes Neumann-conditions, then by the divergence theorem $\int u \cdot n dS = 0$ has to be fulfilled for a solution to exist. Physically this means that there can only exist a steady state when there is no outlow or inflow in the domain.

3 The Wave Equation

Suppose we have a string and let $u(x,t)$ denote the vertical displacement of the string at position x at time t. Furthermore let $T(x,t)$ denote the tension of the string at x,t. Then if we consider a small segment we get due to $F = ma$ the equality

$$\rho_0(x)\Delta x\partial_{tt}u(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + \rho_0(x)\Delta xQ(x,t),$$

where $\theta(x,t)$ denotes the angle of the string at x,t, $\rho_0(x)$ the mass per length and $Q(x,t)$ body forces per unit mass. But then we can use the approximation $\partial_x u \approx \sin\theta$ and let Δx tend to 0 to get

$$\rho_0\partial_{tt}u = \partial_x(T\partial_x u) + \rho_0 Q.$$

If we assume $Q = 0$ and $T = T_0$ constant (i.e. perfect elasticity, meaning T only depends on the local stretching, and θ small) we get the wave equation $u_{tt} = c^2 u_{xx}$ with $c^2(x) = \frac{T_0}{\rho_0(x)}$. If the string has uniform density c is a constant. In higher dimensions the wave equation takes the form $u_{tt} = c^2\Delta u$.

Here we can also impose different boundary conditions.

- If the string is fixed we have $u = \alpha$ at the boundary.
- If the end is free, meaning that it can move along a frictionless vertical track, we can impose $u_x = 0$ as the limiting case $k \to 0$ of the next BC.
- Let's suppose the left end of the string is attached to a spring-mass system obeying Hook's law with equilibrium u_E, meaning $m\ddot{y} = -k(y-u_E) + T_0\partial_x u(0,t)$. Note that the sign of the T_0-term switches at the other end. Then letting $m \to 0$ gives us with $y = u$ the BC $T_0 u_x = k(u - u_E)$ at 0. This corresponds to Newton's law of cooling and imposes the same physical sings.

4 Separation of Variables

For some problems we can separate the variables meaning we can assume a product form solution $u(x,t) = f(x)g(t)$. Then applying differentiation only affects f or g depending on w.r.t. to which coordinate we differentiate. Then we can put everything depending on x on one side and everything depending on t on the other side. Since both terms are equal they have to be constant. Thus we get a problem for x and a problem for t. Then we can solve the BVP (which is mostly in x) first and then the IVP which is mostly in t. Then we get a product solution. Using the principle of superposition any weighted sum of these product solutions satisfies the PDE and the initial condition can be used to determine the weights. Separation of variables can be used several times in higher dimensions when the domain is nice.

5 Fourier-Series

Suppose we have a function f on an interval $[-L,L]$. Then the Fourier-series of f is defined as

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with Fourier-coefficients

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.
\end{aligned}
$$

Fourier's theorem states that when f is pw. smooth then the Fourier-series converges to the periodic extension of f at its continuity points and to $\frac{1}{2}[f(x^+)+f(x^-)]$ else. In this case the Fourier-series is continuous iff f is continuous and additionally $f(-L)=f(L)$. If we are now on the interval $[0,L]$ then we can extend f to a function on $[-L,L]$ for example in an odd or even way. In these cases the Fourier-coefficients a_n/b_n respectively are 0 and thus we obtain a sine/cosine-series instead. The remaining integrals reduce to integrals from 0 to L by symmetry.

The Fourier-series of f can be differentiated term-by-term when it is continuous and f' is pw. smooth. The Fourier series of a function $u(x,t)$ can be differentiated term-by-term w.r.t t if the Fourier-coefficients are pw. smooth in t.

6 Eigenfunction Expansion

To satisfy a PDE $Lu=f$ with initial conditions and homogeneous BC one can expand u in terms of eigenfunctions $L\phi_n=\lambda_n\phi_n$. This will give an ODE for the Fourier-coefficients $a_n(t)$, which can be solved. Alternatively (and also possible when there are nonhomogeneous BC) one can integrate for Fourier-coefficients and then apply Green's identity to retrieve an ODE for the Fourier-coefficients.

7 Sturm-Liouville Problems

A Sturm-Liouville problem is the BVP

$$
(p\phi_x)_x + q\phi + \lambda\sigma\phi = 0
$$

on $[a,b]$ with a homogeneous Robin-BC. It is called regular when additionally $p,\sigma>0$, also at a,b. Then we have that all eigenvalues are real and form a sequence converging to ∞. The eigenfunctions are unique (not necessarily in the periodic case for example), complete with convergence as for Fourier-series and orthogonal w.r.t. σ. Also the so-called Rayleigh-quotient

$$
\lambda = \frac{[-p\phi\phi_x]_a^b + \int_a^b p(\phi_x)^2 - q\phi^2\,dx}{\int_a^b \phi^2\sigma\,dx}
$$

is fulfilled for eigenvalues λ and corresponding eigenfunctions ϕ. To see many of these properties the following considerations can be useful. If $L=\partial_x p\partial_x + q$ is the differential operator from the Sturm-Liouville problem then we get

$$
uLv - vLu = u(pv_x)_x - v(pu_x)_x = (u(pv_x))_x - (v(pu_x))_x = (p(uv_x - vu_x))_x,
$$

which is called Lagrange's identity. Integrating it yields Green's formula, namely

$$\int_a^b uL(v) - vL(u)\,dx = [p(uv_x - vu_x)]_a^b.$$

The rhs vanishes if we have a regular Sturm-Liouville problem or periodic BC (with weight p for the derivative) or a singularity condition and a regular condition at the other end. In this case L is called self-adjoint, since $\int_a^b L(u)v dx = \int_a^b L(v)u dx$. The adjoint operator in general is an operator L^* satisfying

$$\int_a^b uL(v) - vL^*(u)\,dx = [F(u,v)]_a^b.$$

Boundary conditions, which let the rhs vanish are called adjoint boundary conditions.

Green's formula lets us conclude immediately orthogonality of eigenfunctions and real eigenvalues. The Rayleigh quotient can be easily derived by multiplying the PDE with ϕ, integrating both sides and using an integration by parts argument. Furthermore, the smallest eigenvalue is the minimal value of the Rayleigh quotient. This can be used to show that eigenvalues are non-negative or to give an upper bound for the smallest one.

8 The Helmholtz Equation

The higher dimensional analog of the previous section is given by the so-called Helmholtz-equation $\Delta\phi = \lambda\phi$ s.t. $a\phi + b\nabla\phi \cdot n = 0$. Similar statements hold: For example $\lambda \in \mathbb{R}$ and the sequence of EV converges to ∞. But unlike before there may be different EF to the same EV. The EF are complete again and orthogonal, if they belong to different EV. Also the Rayleigh quotient holds:

$$\lambda = \frac{-\int \phi\nabla\phi \cdot n\,dS + \int |\nabla\phi|^2\,dx}{\int \phi^2\,dx}.$$

In nice domains this problem can be treated by applying separation of variables multiple times but this is not always possible. Analogously to the one-dimensional case we can calculate

$$\nabla \cdot (u\nabla v) = u\Delta v + \nabla u \cdot \nabla v, \quad \text{and} \quad \nabla \cdot (v\nabla u) = v\Delta u + \nabla v \cdot \nabla u,$$

which yields again Lagrange's identity

$$\nabla \cdot (u\nabla v - v\nabla u) = u\Delta v - v\Delta u.$$

Applying the divergence theorem gives us Green's formula

$$\int u\Delta v - v\Delta u\,dx = \int u\nabla v - v\nabla u\,dS.$$

Orthogonality of EF and $\lambda \in \mathbb{R}$ can be shown with the same techniques as in one dimension.

9 Non-Homogeneous Problems

There are several tricks to solve non-homogeneous problems. To get rid of constant, non-homogeneous BC one can for example subtract the steady state solution and reduce to homogeneous BC. If the BC are not constant one can subtract any function that does fulfill the BC. This will lead to new inhomogeneities in the equation but removes the non-zero BC.

Now If we have homogeneous BC but sources or sinks one can use the method of eigenfunction expansion. This will lead to ODEs for the Fourier-coefficients which have to be solved. The ODE will also contain the Fourier-coefficients of the source-function. The homogeneous BC are necessary to justify differentiation of the Fourier-series.

To avoid this we can use a similiar approach using Green's identity. Suppose we have eigenfunctions $\phi_n'' + \lambda_n \phi_n = 0$ with homogenous Dirichlet conditions and we're looking for a solution to $u_t = k u_{xx} + Q$, $u(0, \cdot) = A$, $u(L, \cdot) = B$ with some IC. Then we can conclude for the Fourier-coefficients b_n of u

$$b_n' = q_n + \frac{\int_0^L k u_{xx} \phi_n \, dx}{\int_0^L \phi_n^2 \, dx} = q_n + \frac{-\int_0^L \lambda_n k u \phi_n \, dx - [\phi_n'(L)B(t) - \phi_n'(0)A(t)]}{\int_0^L \phi_n^2 \, dx}.$$

But since on the RHS b_n appeared this implies

$$b_n' + k\lambda_n b_n = q_n + \frac{[\phi_n'(0)A(t) - \phi_n'(L)B(t)]}{\int_0^L \phi_n^2 \, dx}.$$

This can be solved by the variation of parameters.

10 Green's Functions for Time-Independent Problems

Given a ODE $L(u) = f$ with some BC for some Sturm-Liouville operator L. Then the corresponding Green's function is given as the solution to $L(G(x, x_s)) = \delta(x - x_s)$ with homogeneous BC. Then the solution u to the original problem can be determined using Green's formula to be

$$u(x_0) = \int_a^b u(x)\delta(x - x_0) \, dx = \int_a^b G(x, x_0) f(x) \, dx + [p(uG_x(x, x_0))]_a^b,$$

where p is from L. The Maxwell reciprocity states now that $G(x_1, x_2) = G(x_2, x_1)$ for all x_1, x_2 and can be proven using Green's formula.

11 How to determine a Green's Function

One can determine a Green's function directly from the defining differential equation. If we consider for example $L = \partial_{xx}$ with Dirichlet-conditions at 0 and L, then we get immediately $\partial_x G(x, x_0) = H(x - x_0) + a$. Integrating another time yields

$$G(x, x_0) = \begin{cases} ax + b & x < x_0 \\ (a+1)x + c & x > x_0 \end{cases},$$

where we can use that G should be continuous at x_0 to obtain $ax_0 + b = (a+1)x_0 + c$. This lets us conclude that $b = x_0 + 1$ and thus we can insert this to get

$$G(x, x_0) = \begin{cases} ax + b & x < x_0 \\ (a+1)x + b - x_0 & x > x_0 \end{cases}.$$

Now we can use the BC to obtain $b = 0$ as well as $a = \frac{x_0 - L}{L}$. This lets us write down the solution as

$$G(x, x_0) = \begin{cases} \frac{x_0 - L}{L} x & x < x_0 \\ \frac{x_0}{L} x - x_0 & x > x_0 \end{cases}.$$

Alternatively, one can also use an eigenfunction expansion to determine G. For that let L be an arbitrary Sturm-Liouville operator and ϕ_n, λ_n corresponding eigenfunctions and eigenvalues. Then if u should satisfy $L(u) = f$ we can conclude for the Fourier-coefficients a_n

$$f = L(u) = \sum_{n=1}^{\infty} -a_n \sigma \lambda_n \phi_n \quad \Rightarrow \quad a_n = \frac{\int_a^b f(x_0)\phi_n(x_0)\, dx_0}{-\lambda_n \int_a^b \phi_n^2 \, d\tilde{x}}$$

under the assumption $\lambda_n \neq 0$. Let's write out u as

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{\int_a^b f(x_0)\phi_n(x_0)\, dx_0}{-\lambda_n \int_a^b \phi_n^2 \, d\tilde{x}} \phi_n(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\phi_n(x_0)\phi_n(x)}{-\lambda_n \int_a^b \phi_n^2 \, d\tilde{x}} \, dx_0$$

Thus, the sum under in the integral at the RHS gives us the Green's function.

12 Fredholm Alternative

If 0 is an eigenvalue of the differential operator it may happen that a non-homogeneous problem has no solution. To investigate this let u fulfill homogeneous BC, such that we can differentiate under the sum to retrieve

$$L(u(x)) = \sum_{n=1}^{\infty} -a_n \sigma \lambda_n \phi_n(x)$$

for $u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$. This will always be orthogonal to the EF corresponding to the 0 EV. Also, if f is orthogonal to that EF, then we can add arbitrary multiples of that EF to u without changing $L(u)$. Consequently, we have either 0 or ∞ many solutions depending on f being orthogonal to the 0-EF or not. This is called the Fredholm alternative.

13 Generalized Green's Functions

If 0 is an eigenvalue Green's functions may not exist due to the considerations in the former section. We will investigate how to avoid this problem. Let's assume $\phi(x_0) \neq 0$, then we can't solve $L(G(x, x_0)) = \delta(x - x_0)$. Let's instead solve $L(G(x, x_0)) = \delta(x - x_0) + c\phi(x)$, where c gets chosen such that $\int (\delta(x - x_0) + c\phi(x))\phi(x)\, dx = 0$. Then there exist solutions and it can be shown that we can choose a solution, which still fulfills $G(x, x_0) = G(x_0, x)$. If we do so we will get again

$$u(x) = \int_a^b f(x_0)G(x, x_0)\, dx_0$$

as a solution to $L(u) = f$ (if f is orthogonal to ϕ).

14 Green's Functions for Poisson's Equation

Let's consider the problem $\Delta G(x, x_0) = \delta(x - x_0)$ with homogeneous Dirichlet conditions. It can be determined using eigenfunction expansion. Then if we are interested in the problem $\Delta u = f$ and $u = h$ on the boundary we can apply Green's formula to obtain

$$u(x) = \int u(x_0)\delta(x - x_0)\, dx_0 = \int f(x_0)G(x, x_0)\, dx_0 + \int h(x_0)\nabla_{x_0} G(x, x_0) \cdot n\, dS$$

On the whole domain we can give a more explicit description with the following considerations. We expect $G(x, x_0)$ to be radially symmetric around x_0 and write it therefore $G(x, x_0) = G(r)$ with $r = |x - x_0|$. Thus since $\Delta G = 0$ for $r \neq 0$ we obtain that $\frac{1}{r}\partial_r(r\partial_r G) = 0$ (d=2). Integrating this twice yields $G(r) = c_1 \log(r) + c_2$. We can assume $c_2 = 0$ for convenience. Using that the integral over any circle must be 1 we get by the divergence theorem

$$1 = \int \Delta G(x, x_0)\, dx = \int \nabla G(x, x_0) \cdot n\, dx = 2\pi r \partial_r G(r) = 2\pi c_1,$$

meaning $c_1 = \frac{1}{2\pi}$. Analogous considerations for $d = 3$ give us $G(x, x_0) = \frac{-1}{4\pi r}$. This allows us now to construct Green's functions on other domains. For example if we impose Dirichlet conditions on a half-space we can just add another point-charge of the opposite sign there. For Neumann-conditions one can use the same sign of charge. We can do the same thing for a circle by putting an opposite charged point-charge on on the line connecting the midpoint and x_0 and adding some constant. This ansatz will give us the Green's function

$$G(x, x_0) = \frac{1}{2\pi} \log\left(\frac{a|x - x_0|}{r_0|x - x_0^*|}\right)$$

with $r_0 = |x_0|$, a the radius of the circle and $x_0^* = \frac{a^2}{r_0}x_0$.

15 Fourier-Transforms

The Fourier-Transform of a function $f(x)$ is defined by $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$. The inverse Fourier-Transform is given by $f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-ix\omega}d\omega$. A table can be found on page 483. The Fourier-transform can be used on the whole domain to transform a PDE into an ODE. In the Fourier-domain this can be solved and transformed back, which gives the solution to the original problem. The method of images can be used to get results on domains with suitable BC. We can also introduce the Fourier-Transform in dimension 2, which takes the form $F(\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(x)e^{i\omega\cdot x}dx$ and its inverse is given by $f(x) = \int_{\mathbb{R}^2} F(\omega)e^{-i\omega\cdot x}d\omega$. A table is on page 513.

16 Time-dependent Green's Functions

If we have a differential operator (also in time) like $L = \partial_t - k\Delta$ or $L = \partial_{tt} - c^2\Delta$ with some BC and initial condition the corresponding Green's function is the solution to $L(G(x,t,x_0,t_0)) = \delta(x-x_0)\delta(t-t_0)$ with homogeneous BC. It describes the influence of a source at x_0, t_0 at time x, t. Thus we can demand the causality principle, meaning $G(x,t,x_0,t_0) = 0$ when $t < t_0$. Also we demand the translation property, meaning $G(x,t,x_0,t_0) = G(x,t-t_0,x_0,0)$.

For the heat equation we can apply Fourier-methods on the whole domain to get that the solution to $u_t = k\Delta u$ with initial condition f on $\mathbb{R}^d$ is given by

$$G(x,t,x_0,t_0) = \left(\frac{1}{4\pi k(t-t_0)}\right)^{\frac{d}{2}} e^{\frac{-|x-x_0|^2}{4k(t-t_0)}}$$

for $t > t_0$. On domains a Green's function can be determined using the method of images. For the wave equation we get first for $L = \partial_{tt} - c^2\Delta$ Green's formula for the wave equation

$$\int_{t_1}^{t_2}\int uL(v) - vL(u)\ dx\,dt = \int \left[u\partial_t v - v\partial_t u\right]_{t_1}^{t_2}\ dx - c^2\int_{t_1}^{t_2}\int (u\nabla v - v\nabla u)\cdot n\,dS\,dt$$

by applying Green's formula to the time and spatial part separately. With that we can derive a formula to construct a solution u to the wave equation with BC, IC and sources. Similar things can be done for the heat equation only that the operator is not self-adjoint since it contains only one time-derivative.

17 D'Alemberts Formula

Let's investigate the wave equation $u_{tt} = c^2u_{xx}$ on $\mathbb{R}$. If we introduce $w = u_t - cu_x$ and $v = u_t + u_x$ then these functions fulfill the first order PDEs $w_t + cw_x = 0$ and $v_t - cv_x = 0$. It is easy to see that $w = P(x-ct)$ and $v = Q(x+ct)$ solve these equations. Consequently, we get

$$2u_t = P(x-ct) + Q(x+ct) \quad \text{and} \quad 2cu_x = Q(x+ct) - P(x-ct),$$

which gives us $u(x,t) = F(x-ct) + G(x+ct)$ by $cF' = -\frac{P}{2}$ and $cG' = \frac{Q}{2}$. Note that integration constants can be hidden in and F or G. The sets on which $x - ct$ resp. $x + ct$ are constant are called the characteristics of u since F and G travel along these. Now if we impose IC $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ we can use our given formula to determine F and G, which gives us the formulas

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} \quad \text{and} \quad G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}.$$

Inserting this in our formula for u gives us D'Alembert's solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}.$$

If we want to use this result on a domain we have to be aware that our formulas for F and G only work on the domain. Outside of this F and G are still interesting since $x \pm ct$

can be arbitrary. One way to solve this is to use the boundary conditions to get a relation between $F$and G which will give us then F and G outside of the domain. Alternatively, one can also work with even or odd periodic extensions of f and g. If chosen properly the solution will fulfill the desired BC.

18 The Method of Characteristics

Assume we have a first order PDE $au_t + bu_x = c$. Then if a curve $(t(s), x(s), u(s))$ fulfills

$$\begin{cases} t'(s) = a(x,t,u) \\ x'(s) = b(x,t,u) \\ u'(s) = c(x,t,u) \end{cases} .$$

and its initial value is on the graph of the solution it will lie entirely on the solution graph. We can choose for example $s = 0$ and $t = 0$ under the assumption that the time-domain begins in 0 as well. We can also parametrize our initial function f by some parameter λ. Then for each initial value $s_0 = 0$, $t_0 = 0$, $x_0 = \lambda$ and $u_0 = f(\lambda)$ we will get a curve in s. Thus we will have a plane in (λ, s). When we can retrieve (λ, s) from (x, t) we found our solution to the PDE.

19 Traffic Flow

Let's assume we have a road with car density ρ and car velocity $u(\rho)$. Then the traffic flow is given by $q(\rho) = \rho u(\rho)$. Conservation of cars reads then as

$$\partial_t \int_a^b \rho(x) dx = q(a,t) - q(b,t),$$

which means $\partial_t \rho + \partial_x q = 0$. If we define $c(\rho) = \partial_\rho q(\rho)$, then we retrieve the quasilinear PDE $\rho_t + c(\rho)\rho_x = 0$. Note that if only the PDE is given q can be be obtained as the anti-derivative of c, which will be important later when considering shock waves. To solve the traffic flow problem we can apply the method of characteristics.

Let's consider for example the problem $\rho_t + 2\rho\rho_x = 0$ with IC

$$\rho(x,0) = \begin{cases} 3, & x < 0 \\ 4 & x > 0 \end{cases} .$$

Then the characteristics fulfill $x'(t) = 2\rho$. We impose the initial conditions $t_0 = 0$, $x_0 \in \mathbb{R}$ arbitrary and $\rho_0 = \rho(x_0, 0)$ correspondingly. The solution is given by $x(t) = x(0) + 2\rho_0 t$ since ρ is constant along these characteristics. We observe that at $(x, t) = (0, 0)$ there is some space which is not filled by characteristics since right of it the characteristics travel faster than left of it. The characteristics, which we will find in this space have to suffice $x = 2\rho t$, which gives us $\rho = \frac{x}{2t}$ there. This is called fanlike characteristics.

If we consider the same problem with switched initial condition

$$\rho(x,0) = \begin{cases} 4, & x < 0 \\ 3 & x > 0 \end{cases}$$

the characteristics on the left are faster than the ones on the right instead. Thus a shock wave will appear. The shock velocity is given by

$$\frac{q(x_s^+,t)-q(x_s^-,t)}{\rho(x_s^+,t)-\rho(x_s^-,t)}.$$

We know that the denominator here is -1 already. Since $q(\rho) = \frac{\rho^2}{2}$, we have in the enumerator -7 and in this case the shock wave travels with speed 7 to the right.

20 Laplace-Transforms

Similarly to the Fourier-Transform the Laplace-Transform transforms differential operations to algebraic operations and can thus be used to solve PDEs. Unlike the Fourier-transform the Laplace-transform acts on functions defined on $[0,\infty)$ and is thus usually applied in the time variable. It is given by

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt.$$

A table with its properties can be found on page 612.

21 Uniqueness Proofs

Uniqueness of solutions for the heat and wave equation can be shown using energetic methods. For the heat equation we can introduce

$$E(t) = \frac{1}{2}\int u^2\,dx.$$

Differentiating, using the heat equation and the divergence theorem gives us that E decays in time, meaning that solutions are unique. Analogously for the wave equation we can define

$$E(t) = \frac{1}{2}\int c^2|\nabla u|^2 + u_t^2\,dx.$$

Differentiating, using the divergence theorem and applying the wave equation shows that E is constant in time, which gives uniqueness.

22 Interesting Examples

- One dimensional BVPs and properties p.66: Classification of resulting eigenfunctions for a one-dimensional BVP.
- Laplace equation on a rectangle p.68: Split the problem in a problem with only one non-zero BC and solve each of them separately. Then add them up.

- Laplace equation on a circular domain p.73: The circular Laplacian is given by $\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u$. The initial resp. boundary conditions consist of a periodicity condition along θ, a boundedness condition at $r = 0$ and a given function along the boundary of the circle. The radial problem is of the form $r^2G''(r) + rG'(r) - \lambda G(r) = 0$ and gets solved by rational functions (and the logarithm for $\lambda = 0$). The boundedness condition tells us what G is appropriate.

- Example with Robin-BC p.201: We can't compute the eigenvalues directly so we approximate them with graphical techniques and get estimates and approximations for them.

- Forced membranes p. 374: Apply the method of eigenfunction expansion to the wave equation in 2 dimensions with a source term Q. If the frequency of a periodic forcing Q coincides with the frequency of a mode of the membrane then the Fourier-coefficient grows to infinity.

- Poisson's equation on a rectangular domain p.382: We can decompose Poisson equation with non-homogeneous BC and source terms in a Laplace equation and a Poisson equation with homogeneous BC. The first one was solved above for simple geometries. For the latter one, one can expand it in terms of the eigenfunctions and gets then equations on the Fourier-coefficients. Instead, one can also try to tackle both at once using Green's formula for $\Delta f = Q$, $f = \alpha$ on the boundary:

$$\int -f\lambda_n\phi_n \, dx = \int Q\phi_n \, dx + \int \alpha\nabla\phi_n \cdot n \, dS.$$

- Green's function for a circle p.443: Using the ansatz

$$G(x, x_0) = \frac{1}{2\pi}\log(|x - x_0|) - \frac{1}{2\pi}\log(|x - x_0^*|) + c$$

  with $x_0^* = \gamma x_0$ yields proper constants γ and c using the BC.

- Application of the Laplace-Transform to the wave equation p.623: The Laplace transform yields an ODE in space which can be solved using the BC as initial condition. Transforming back via the table gives us the solution.

23 Things to know by Heart

The Robin-BC

$$u'(0) = Hu(0) \quad \text{and} \quad u'(L) = -Hu(L)$$

has in physical cases the sign $H > 0$. The regular Sturm-Liouville problem is given by

$$(p\phi_x)_x + q\phi + \lambda\sigma\phi = 0$$

with Robin-BC and $p, \sigma > 0$ on the closed interval. Green's Formula reads as follows

$$\int_a^b uL(v) - vL(u) \, dx = [p(uv_x - vu_x)]_a^b,$$

where $L = \partial_x(p\partial_x) + q$ for this problem. In higher dimensions we have Green's formula given by

$$\int u\Delta v - v\Delta u \, dx = \int u\nabla v - v\nabla u \, dS.$$

The Green's function for the Poisson equation is given by $\frac{1}{2\pi}\log(r)$ in $d = 2$ and $\frac{-1}{4\pi r}$ for $d = 3$ with $r = |x - x_0|$. D'Alemberts form for solutions of the wave equation is given by $u(x,t) = F(x - ct) + G(x + ct)$. The shock velocity for traffic flow models is given by

$$\frac{q(x_s^+, t) - q(x_s^-, t)}{\rho(x_s^+, t) - \rho(x_s^-, t)}.$$

24 Further tricks

Variation of Constants: Suppose we want to solve the problem $f'(t) + \lambda f(t) = g(t)$. Then we van multiply by $e^{\lambda t}$ to get $\partial_t(e^{\lambda t} f(t)) = e^{\lambda t} g(t)$. Then we can integrate.