Resit Exam Mathematical Structures TW1010
Thursday April 18, 2019, 9:00-12:00

TU Delft

No calculators allowed. Write the solutions in the fields provided. The grade is (score+8)/8.

Exercise [ ] continued (extra space)

Exercise 1 is at the bottom of this page!

Exercise [ ] continued (extra space)

1 Determine using a truth table whether or not $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology. 4

The truth table is given by

- [ ] The statement is a tautology, since
- [ ] The statement is not a tautology, since

Don't forget Exercise 1 on the previous page!

2 Give a relation on $\mathbb{Q}$ which is transitive, reflexive, but not symmetric. 8

The relation R is defined as xRy holds whenever

The relation R is reflexive as

The relation R is not symmetric as

The relation R is transitive as

3 Find the error in the following proof. 4

Theorem: For any function $f : \mathbb{R} \to \mathbb{R}$ we have $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

Proof:

1. Suppose $y \in f(A \setminus C)$.
2. Then there exists $x \in A \setminus C$ with $f(x) = y$.
3. Therefore $x \in A$ and $x \notin C$.
4. As $x \in A$ we have $f(x) \in f(A)$.
5. As $x \notin C$ we have $f(x) \notin f(C)$.
6. Hence $f(x) \in f(A) \setminus f(C)$.
7. As $y = f(x)$ we conclude $y \in f(A) \setminus f(C)$.
8. As we have shown for all y that $y \in f(A \setminus C) \Rightarrow y \in f(A) \setminus f(C)$ we have $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

The error in the proof occurs at line number ☐. This statement is wrong as

4 Formulate the completeness axiom for the real numbers. 2

5a Suppose $f : \mathbb{R} \to \mathbb{R}$ is a decreasing function and $A \subseteq \mathbb{R}$ is a bounded set. Show that 6

$$\sup(f(A)) \leq f(\inf(A)).$$

5b Give a decreasing function $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subseteq \mathbb{R}$ for which the strict inequality $\sup(f(A)) < f(\inf(A))$ holds; and show that your example works. 4

6 Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, show that if $x, y \in \mathbb{R}$ with $x < y$, then there is a rational number $q \in \mathbb{Q}$ with $x < q < y$. 6

7 The sequence (a_n) is defined recursively as $a_{n+1} = \sqrt{8 + \frac{1}{2}a_n a_{n-1}}$, starting with $a_1 = 1$ and $a_2 = 2$.

7a Use induction to prove that (a_n) is increasing. 7
Hint: Use the statement $P(n) : a_n \leq a_{n+1} \leq a_{n+2}$.

7b We still use the sequence (a_n) defined by $a_{n+1} = \sqrt{8 + \frac{1}{2} a_n a_{n-1}}$, $a_1 = 1$, and $a_2 = 2$.

Show that (a_n) converges. 6

7c Determine the limit $\lim a_n = a$. 2

8 Suppose $f(x)$ satisfies $|f(x) - f(y)| \leq \sqrt{|x-y|}$. Suppose (s_n) is a Cauchy sequence, show that $(f(s_n))$ is also a Cauchy sequence using the definition of Cauchy sequence. 6

The axioms of an ordered field as applied to $\mathbb{R}$ are

A1 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x + y = w + z$;
A2 $\forall x, y \in \mathbb{R} : x + y = y + x$;
A3 $\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$;
A4 $\exists 0 : \forall x \in \mathbb{R} : x + 0 = x$ and this 0 is unique;
A5 $\forall x \in \mathbb{R} : \exists (-x) \in \mathbb{R} : x + (-x) = 0$ and $(-x)$ is unique;
M1 $\forall x, y \in \mathbb{R} : x \cdot y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x \cdot y = w \cdot z$;
M2 $\forall x, y \in \mathbb{R} : x \cdot y = y \cdot x$;
M3 $\forall x, y, z \in \mathbb{R} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
M4 $\exists 1 \neq 0 : \forall x \in \mathbb{R} : x \cdot 1 = x$ and this 1 is unique;
M5 $\forall x \neq 0 : \exists (1/x) \in \mathbb{R} : x \cdot (1/x) = 1$ and $(1/x)$ is unique;
DL $\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = x \cdot y + x \cdot z$;
O1 For all $x, y \in \mathbb{R}$ exactly one of $x = y$, $x > y$, holds $x < y$;
O2 $\forall x, y, z \in \mathbb{R} : x < y \wedge y < z \Rightarrow x < z$;
O3 $\forall x, y, z \in \mathbb{R} : x < y \Rightarrow x + z < y + z$;
O4 $\forall x, y, z \in \mathbb{R} : x < y \wedge 0 < z \Rightarrow xz < yz$.

9 Show using the axioms that $(x + y)^2 = x^2 + (2(xy) + y^2)$. 6
Here we use the notations $x^2 = x \cdot x$ and $2 = 1 + 1$.
Be sure to precisely indicate what axioms you use in each step.

10 Give the definition of convergence of a series. A series $\sum_{n=1}^{\infty} a_n$ converges if 2

11 Determine whether or not the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges, and if it does converge, whether this convergence is absolute or conditional. 4

12 Determine for all x whether $\sum_{n=1}^{\infty} \frac{(2n)!}{n!} x^n$ converges or diverges. Also determine when the series is absolutely or conditionally convergent. 5

(Fill in things like $x \in [2, 3)$ or $x = 5$ in the boxes below after doing your calculations.)

- The series converges absolutely for
- The series converges conditionally for
- The series diverges for

Examiner responsible: Fokko van de Bult
Examination reviewers: Wolter Groenevelt, Rik Versendaal.