Midterm Mathematical Structures TW1010
Monday November 5, 2018, 9:00-11:00

No calculators allowed. Write the solutions in the fields provided. The grade is (score+6)/6.

1 For each of the following, prove or give a counterexample

2

1a $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : x > y$

Solution. This is true. Let $x \in \mathbb{R}$ be arbitrary. Take $y = x - 1$. Then $x > x - 1 = y$. □

2

1b $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R} : x > y$

Solution. This is false. The negation is $\forall y \in \mathbb{R} \exists x \in \mathbb{R} : x \leq y$. Let $y \in \mathbb{R}$ be arbitrary. Choose $x = y$, then $x \leq y$ holds. □

2 Give the logical tautology which underlies the structure of the following proof. Also be specific which statements the variables p, q, etc. represent. **Theorem:** If n is a prime number, then either $n = 2$ or n is odd. 3

Proof: Suppose n is a prime number which is not odd. Then n is even, so n is a divisible by 2. As any prime number is divisible only by 1 and itself, n must equal 2. Thus we have shown that any prime number is either 2 or odd.

Solution. The statement is $p \Rightarrow q \vee r$, where

- p is the statement: "n is a prime number"
- q is the statement: "$n = 2$"
- r is the statement: "n is odd".

The tautology used is

$$(p \Rightarrow q \vee r) \quad \Leftrightarrow \quad (p \wedge \sim r \Rightarrow q)$$

□

3 Give an example of a relation on $\mathbb{N}$ which is reflexive, symmetric, but not transitive. 7

Be sure to show the example you describe satisfies these properties.

Solution. Take as relation nRm if $|n - m| \leq 1$.

- Reflexivity: Let n be arbitrary. As $|n - n| = 0 \leq 1$ we indeed have nRn.
- Symmetry: Let n, m be arbitary and assume nRm. Then $|n - m| \leq 1$, thus $|m - n| = |n - m| \leq 1$ too. Therefore mRn holds as well.
- Transitivity: It is not transitive as $1R2$ and $2R3$, but not $1R3$ (indeed $|1-2| = 1 \leq 1$, $|2 - 3| = 1 \leq 1$, but $|1 - 3| = 2 > 1$).

□

4 Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = 5x + 4$. Show that f is injective. 4

Solution. Suppose $f(x_1) = f(x_2)$. Then $f(f(x_1)) = f(f(x_2))$, thus $5x_1 + 4 = 5x_2 + 4$. It follows that $5x_1 = 5x_2$, so $x_1 = x_2$. We have shown that f is injective. □

5 Given a function $f : A \to B$ and a subset $C \subseteq A$.

5a Complete the definition: The image $f(C)$ equals 2

Solution. the set $\{f(x) \ : \ x \in C\}$.
Alternatively: the set $\{y \in B \ : \ \exists x \in C : f(x) = y\}$.
Alternatively: the set of all images under f of elements from the set C. □

5b Show that $f(f^{-1}(f(C))) = f(C)$. 5

Solution. This is an equality between sets. Assume $y \in f(f^{-1}(f(C)))$. Then there is a $x \in f^{-1}(f(C))$ with $y = f(x)$. By definition of inverse image this means that indeed $y \in f(C)$. Therefore $f(f^{-1}(f(C))) \subseteq f(C)$.
Now suppose $y \in f(C)$. Then there is an $x \in C$ such that $f(x) = y$. Thus for this x we know $x \in f^{-1}(f(C))$. But then $y = f(x) \in f(f^{-1}(f(C)))$. Therefore $f(C) \subseteq f(f^{-1})f(C)))$.
As we have shown both sides are subsets of each other we can conclude $f(f^{-1}(f(C))) = f(C)$. □

6 Define the sequence (F_n) recursively by $F_1 = 1$ and $F_2 = 1$ and $F_{n+2} = 2F_{n+1} + 3F_n$. 9
Show that for all $n \in \mathbb{N}$ we have

$$F_1 + F_2 + F_3 + \cdots + F_n = \frac{1}{4}(F_{n+2} - F_{n+1})$$

Solution. We prove this using induction to n.

Note that $F_3 = 2F_2 + 3F_1 = 5$. For $n = 1$ we thus have that $F_1 = 1$ and $\frac{1}{4}(F_3 - F_2) = \frac{1}{4}(5 - 1) = 1$ as well. The equality is thus true for $n = 1$.

Now assume $F_1 + F_2 + \cdots + F_k = \frac{1}{4}(F_{k+2} - F_{k+1})$ for some k. Then we have

$$\begin{aligned} F_1 + F_2 + \cdots + F_k + F_{k+1} &= \frac{1}{4}(F_{k+2} - F_{k+1}) + F_{k+1} = \frac{1}{4}F_{k+2} + \frac{1}{4} \cdot 3F_{k+1} \\ &= \frac{1}{4}F_{k+2} + \frac{1}{4}(F_{k+3} - 2F_{k+2}) = \frac{1}{4}(F_{k+3} - F_{k+2}). \end{aligned}$$

Thus we see that the equation also holds for $n = k + 1$.

By induction we have shown that $\sum_{i=1}^{n} F_i = \frac{1}{4}(F_{n+2} - F_{n+1})$ holds for all $n \in \mathbb{N}$. □

7 Show that $\lim \frac{4n^2+n+3}{2n^2-n} = 2$ using the definition of limit of a sequence. 8

Solution. Let $\epsilon > 0$ be arbitary. Choose $N = \max(3, 4/\epsilon)$. Let $n > N$ be arbitary. Then

$$|\frac{4n^2 + n + 3}{2n^2 - n} - 2| = |\frac{3n + 3}{2n^2 - n}| \leq \frac{4n}{n^2} = \frac{4}{n} < \frac{4}{N} \leq \epsilon,$$

where the first inequality holds as $3n + 3 \leq 4n$ for $n \geq 3$ and $2n^2 - n \geq n^2$ for all $n \geq 1$. □

8 Suppose (s_n) is a convergent sequence with $s_n \geq 0$ for all n, and $\lim s_n = s > 0$. Show 6
that $\lim\ (s_n)^{\frac{1}{3}} = s^{\frac{1}{3}}$.

Hint: The equation $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ might be useful.

Remark: You are not allowed to use continuity in your solution.

Solution. Let $\epsilon > 0$ be arbitrary. Choose N such that $|s_n - s| < \epsilon s^{\frac{2}{3}}$ for all $n > N$. Now, let $n > N$ be arbitary. Then

$$|(s_n)^{\frac{1}{3}} - s^{\frac{1}{3}}| = \frac{|s_n - s|}{|(s_n)^{\frac{2}{3}} + (ss_n)^{\frac{1}{3}} + s^{\frac{2}{3}}|} \leq \frac{|s_n - s|}{s^{\frac{2}{3}}} < \frac{\epsilon s^{\frac{2}{3}}}{s^{\frac{2}{3}}} = \epsilon.$$

Thus $\lim (s_n)^{\frac{1}{3}} = s^{\frac{1}{3}}$ as desired. □

9 Show that if (s_n) satisfies $\lim s_n = \infty$, then for any $k \in \mathbb{R}$ we have $\lim(s_n + k) = \infty$ as well. 5

Solution. Let M be arbitary. Then there exists N such that $s_n > M - k$ for all $n > N$. Now let $n > N$ be arbitary. Then $s_n + k > (M - k) + k = M$. Therefore $\lim s_n + k = \infty$. □

10 Complete the definition: The sequence (s_n) is bounded if 2

Solution. there exists $M \in \mathbb{R}$ such that $|s_n| < M$ for all $n \in \mathbb{N}$.

In symbols: $\exists M \in \mathbb{R} : \forall n \in \mathbb{N} : |s_n| < M$. □

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