# Test 2 Mathematical Structures TW1010
# Friday December 14, 2018, 10:45-11:45

No calculators allowed. Write the solutions in the fields provided. The grade is (score+4)/4.

1 The sequence $(s_n)$ is defined recursively as $s_1 = 0$ and $s_{n+1} = \frac{1}{3-s_n}$ for $n \geq 1$.

1a Show that $0 \leq s_n \leq 2$ for all $n \in \mathbb{N}$. 4

*Solution.* We show this by induction. For $n = 1$ we have $s_1 = 0$, so $0 \leq s_1 \leq 2$ is correct. Assume $0 \leq s_k \leq 2$ for some $k$. Using $s_{k+1} = \frac{1}{3-s_n}$ we find that from $0 \leq s_k$ we have $s_{k+1} \geq \frac{1}{3} \geq 0$ and from $s_k \leq 2$ we obtain $s_{k+1} \leq \frac{1}{3-2} = 1 \leq 2$. By induction we have thus shown that $0 \leq s_n \leq 2$ for all $n \in \mathbb{N}$. □

1b Show that the sequence $(s_n)$ converges. 5

*Solution.* We will use the monotone convergence theorem which says that a bounded monotone sequence converges. The sequence is bounded by part a; we will prove that it is increasing. By induction we will show $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. $s_2 = \frac{1}{3-s_1} = \frac{1}{3} \geq s_1$ so this is true for $n = 1$. Now assume $s_k \leq s_{k+1}$. Then $s_{k+2} = \frac{1}{3-s_{k+1}} \geq \frac{1}{3-s_k} = s_{k+1}$ as $\frac{1}{3-x}$ is an increasing function for $x < 3$ and $s_k < 2$ by part a. We conclude that $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$, thus the sequence is increasing. By the monotone convergence theorem we can thus conclude that this monotone bounded sequence converges. □

1c Obtain the limit $\lim s_n$. 3

*Solution.* Using the rules of calculations for limits (applicable as the sequence converges) we observe

$$s = \lim s_n = \lim s_{n+1} = \lim \frac{1}{3 - s_n} = \frac{1}{3 - \lim s_n} = \frac{1}{3 - s}.$$

Thus we find $3s - s^2 = 1$, so $s^2 - 3s + 1 = 0$. This means $s = \frac{3}{2} \pm \frac{1}{2}\sqrt{9-4} = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$. Observe that $\frac{3}{2} + \frac{1}{2}\sqrt{5} > \frac{3}{2} + \frac{1}{2} = 2$, so this is impossible as limit as $s_n \leq 2$ for all $n$. Thus we must have $s = \frac{3}{2} - \frac{1}{2}\sqrt{5}$. □

The axioms of an ordered field as applied to $\mathbb{R}$ are

A1 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x + y = w + z$;
A2 $\forall x, y \in \mathbb{R} : x + y = y + x$;
A3 $\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$;
A4 $\exists 0 : \forall x \in \mathbb{R} : x + 0 = x$ and this 0 is unique;
A5 $\forall x \in \mathbb{R} : \exists (-x) \in \mathbb{R} : x + (-x) = 0$ and $(-x)$ is unique;
M1 $\forall x, y \in \mathbb{R} : x \cdot y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x \cdot y = w \cdot z$;
M2 $\forall x, y \in \mathbb{R} : x \cdot y = y \cdot x$;
M3 $\forall x, y, z \in \mathbb{R} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
M4 $\exists 1 \neq 0 : \forall x \in \mathbb{R} : x \cdot 1 = x$ and this 1 is unique;
M5 $\forall x \neq 0 : \exists (1/x) \in \mathbb{R} : x \cdot (1/x) = 1$ and $(1/x)$ is unique;
DL $\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = x \cdot y + x \cdot z$;
O1 For all $x, y \in \mathbb{R}$ exactly one of $x = y$, $x > y$, holds $x < y$;
O2 $\forall x, y, z \in \mathbb{R} : x < y \wedge y < z \Rightarrow x < z$;
O3 $\forall x, y, z \in \mathbb{R} : x < y \Rightarrow x + z < y + z$;
O4 $\forall x, y, z \in \mathbb{R} : x < y \wedge 0 < z \Rightarrow xz < yz$.

2 Assume $0 < x, y, z, w$. Show using the axioms that if $x < y$ and $w < z$ then $xw < yz$. Be sure to state which axiom you use in each step! 6

*Solution.* From O4 we obtain that $x < y$ and $0 < w$ implies $xw < yw$.

From M2 we then have $yw = wy$, thus $xw < wy$.

From another O4 and $w < z$ and $0 < y$ implies $wy < zy$.

From O2 and $xw < wy$ and $wy < zy$ we obtain $xw < zy$.

Finally another M2 gives $zy = yz$, so $xw < yz$. □

3 Finish the definition. A sequence $(s_n)$ is called a Cauchy sequence if 2

*Solution.* for all $\epsilon > 0$ there is a $N$ such that for all $n, m > N$ we have $|s_n - s_m| < \epsilon$. □

4 Consider the sequence $(s_n)$ defined by $s_n = \cos(\frac{n\pi}{6})$. Determine $\limsup s_n$. Be sure to precisely show that your answer is correct. 8

*Solution.* We will show $\limsup s_n = 1$. Indeed for all $n$ we have $s_n \leq 1$, so for all $\epsilon > 0$, there is a $N$ (namely 1) such that for all $n > N$ we have $s_n < 1 + \epsilon$.

Moreover, suppose $m < 1$ and let $N$ be arbitrary. Take $n = 12N$, then $n > N$ and $s_n = \cos(\frac{12N\pi}{6}) = \cos(2\pi N) = 1 > m$.

Thus we have checked the two properties satisfied by a lim sup. □

5 Given a non-empty, bounded set $S \subseteq (0, \infty)$. Let $T = \{1/s \mid s \in S\}$ be the set of all reciprocals[1] of elements from $S$.

5a Show that $T$ has an infimum 2

*Solution.* Observe that $T$ is a non-empty set (as $S$ is non-empty). Moreover all elements in $T$ are bigger than 0, so it is bounded below. Hence $T$ has an infimum by the completeness axiom. □

5b Show that $\inf(T) = 1/\sup(S)$. 6

*Solution.* We will show that $1/\sup(S)$ equals the infimum of $T$ by checking the properties of the infimum. We use the characterization of infimum from Practice 3.3.6.

- First we show $1/\sup(S)$ is a lower bound to $T$. Let $t \in T$. Then there is $s \in S$ with $t = 1/s$. As $s \leq \sup(S)$ we find that $t = 1/s \geq 1/\sup(S)$. In particular $1/\sup(S)$ is indeed a lower bound to $T$.
- Now suppose $m > 1/\sup(S)$. Then $1/m < \sup(S)$. Thus there is $s \in S$ with $1/m < s$. But then $1/s \in T$ and $1/s < m$. In particular we see that $m$ is not a lower bound to $T$.

We conclude that $1/\sup(S)$ is the greatest lower bound of $T$ and thus the infimum. □

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[1] Reciproval betekent inverse van een getal in het Nederlands