Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
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Final exam Linear Algebra 1, TW1030
Tuesday 30 October 2018, 09:00-12:00.

Name:

Student ID:

This exam consists of two parts: short answer questions and open questions. ***The answer to each open question must contain a complete explanation.*** *Devices like calculators, mobile phones, laptops, sheets with formulas, etc, may not be used. The grade is determined by adding 6 points to your total score and dividing the result by 6.*

Short answer questions (10 questions, 5 on the other side!)

(2) **K1.** We have

$$A = \begin{bmatrix} 1 & 1 & 3 & -2 \\ 2 & -5 & 3 & 1 \\ -3 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & -2 \\ 0 & -7 & -3 & 5 \\ 0 & 4 & 11 & -4 \end{bmatrix} = B.$$

Give a matrix E such that $B = EA$.

(2) **K2.** Circle "yes" if the given transformation is linear, otherwise circle "no".

(a) The map $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_1([x_1, x_2, x_3]^T) = [x_1 - |x_2| + x_3, 2x_1 + x_2 + x_3]^T$ yes/no

(b) The orthogonal projection $T_2 : \mathbb{R}^5 \to \mathbb{R}^5$ on $\text{Nul}(\begin{bmatrix} 3 & -1 & 0 & 2 & 0 \end{bmatrix})$ yes/no

(c) The map $T_3 : \mathbb{R}^4 \to \mathbb{R}$ defined by $T_3([x_1, x_2, x_3, x_4]^T) = \det \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ yes/no

K3. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection on the line $y = x$, followed by the rotation *counterclockwise* about the origin through an angle of 135°.

(2) (a) Give the standardmatrix of T.

($1\frac{1}{2}$) (b) Circle two times "yes" or "no": T is

injective: yes/no
surjective: yes/no

(3) **K4.** A is the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. The rank of this matrix is 3 and the rank of the matrix $A - I$ is less than three. Mark for each of the following statements whether it is true or false.

- 2 is an eigenvalue because each diagonal entry of A equals 2. true/false
- 1 is an eigenvalue because the rank of $A - I$ is smaller than 3. true/false
- 0 is not an eigenvalue because the rank of A is 3. true/false
- the vector $[1, 1, 1]^T$ is an eigenvector of A. true/false

K5. Let $A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & 7 \\ 0 & 0 & -2 \end{bmatrix}$.

(2) (a) The eigenvalues of A^{-1} are

(2) (b) Give an eigenvector of A^{-1} corresponding to a negative eigenvalue.

($1\frac{1}{2}$) **K6.** Circle for each of the following statements about the matrices

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

whether it is true or false:

(a) A is an orthogonal matrix. True/False

(b) B is an orthogonal matrix. True/False

(2) **K7.** Give all possible values for the determinant of an orthogonal matrix.

(2) **K8.** How can the matrix of the orthogonal projection on Col(A) be found if $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

short explanation

(2) **K9.** For $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ and $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ it is given that $\mathbf{p}$ is the orthogonal projection of $\mathbf{b}$ on W. Find a en b.

$a =$ | $b =$

K10. Let $A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & b & a \\ 4 & 8 & 9 \end{bmatrix}$ with $a, b \in \mathbb{R}$.

(2) (a) Find all values of a en b that make this matrix symmetric and positive definite.

a | b

(2) (b) Take $a = 8$ and $b = 18$. Why does there not exist an invertible 3×3-matrix W such that $A = W^T W$?

short explanation

END OF SHORT ANSWER QUESTIONS

TW1030, Tuesday 30 October 2018, overview open questions, to answer on other sheets

1. Let

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Furthermore $A = PDP^{-1}$. Answer the next questions without computing A! Explain your answers!

(2) (a) Find $\det(A)$.

(2) (b) True or false: $A^{-1} = A$?

(3) (c) Let $\mathbf{x}_0 = (0, 0, 3)^T$. For $k \in \mathbb{N}$ find $A^k \mathbf{x}_0$.

(2) (d) Let

$$R = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Are A and RCR^{-1} equal?

(2) (e) Is A symmetric?

(2) (f) Does there exist a matrix B such that $A = B^T B$?

(2) (g) Find $\text{Nul}(A - 3I)$.

2. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 1 & 4 & 3 & 4 & 3 \\ 3 & 4 & 1 & 0 & -3 \\ 1 & -1 & -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

and let W be the nullspace of A. Furthermore let

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

(2) (a) Show that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Do this without solving a system!!

(3) (b) Find the orthogonal projection $\mathbf{p}$ of $\mathbf{b}$ on W and find the distance of $\mathbf{b}$ to W.

(2) (c) Give a vector $\mathbf{c}$, not equal to $\mathbf{b}$ or $\mathbf{p}$, such that its orthogonal projection on W equals $\mathbf{p}$.

3. In $\mathbb{R}^n$ (with $n \geq 2$) are taken two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 1$. With these vectors we define the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\mathbf{x}) = \mathbf{x} - (\mathbf{u} \cdot \mathbf{x})\mathbf{v}$$

(you don't need to prove that T is linear).

(1) (a) Is $\mathbf{v}$ een eigenvector of T? If yes, also give the corresponding eigenvalue.
[REMARK: λ is an eigenvalue of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ if an eigenvector $\mathbf{x} \neq \mathbf{0}$ exists, such that $T(\mathbf{x}) = \lambda \mathbf{x}$.]

(1) (b) (*i*) Show that the nonzero vectors orthogonal with $\mathbf{u}$ are eigenvectors of T. Also give the corresponding eigenvalue.

(2) (*ii*) Why is the dimension of the corresponding eigenspace equal to $n - 1$?

(2) (c) Show, by using (a) and (b), that T is diagonalizable.
[REMARK: a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable if and only if $\mathbb{R}^n$ has a basis containing only eigenvectors of T.]