Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Van Mourik Broekmanweg 6, Delft

Final exam Linear Algebra 1, TW1030
Tuesday 30 October 2018, 09:00-12:00.

Name: Joost

Student ID:

This exam consists of two parts: short answer questions and open questions. ***The answer to each open question must contain a complete explanation.*** *Devices like calculators, mobile phones, laptops, sheets with formulas, etc, may not be used. The grade is determined by adding 6 points to your total score and dividing the result by 6.*

Short answer questions (10 questions, 5 on the other side!)

(2) **K1.** We have

$$A = \begin{bmatrix} 1 & 1 & 3 & -2 \\ 2 & -5 & 3 & 1 \\ -3 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & -2 \\ 0 & -7 & -3 & 5 \\ 0 & 4 & 11 & -4 \end{bmatrix} = B.$$

Give a matrix E such that $B = EA$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(2) **K2.** Circle "yes" if the given transformation is linear, otherwise circle "no".

(a) The map $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_1([x_1, x_2, x_3]^T) = [x_1 - |x_2| + x_3, 2x_1 + x_2 + x_3]^T$ yes/no

(b) The orthogonal projection $T_2 : \mathbb{R}^5 \to \mathbb{R}^5$ on Nul$(\begin{bmatrix} 3 & -1 & 0 & 2 & 0 \end{bmatrix})$ yes/no

(c) The map $T_3 : \mathbb{R}^4 \to \mathbb{R}$ defined by $T_3([x_1, x_2, x_3, x_4]^T) = \det \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ yes/no

K3. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection on the line $y = x$, followed by the rotation *counterclockwise* about the origin through an angle of 135°.

(2) (a) Give the standardmatrix of T.

$$\begin{bmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

($1\frac{1}{2}$) (b) Circle two times "yes" or "no": T is

injective: yes/no
surjective: yes/no

(3) **K4.** A is the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. The rank of this matrix is 3 and the rank of the matrix $A - I$ is less than three. Mark for each of the following statements whether it is true or false.

- 2 is an eigenvalue because each diagonal entry of A equals 2. true/false
- 1 is an eigenvalue because the rank of $A - I$ is smaller than 3. true/false
- 0 is not an eigenvalue because the rank of A is 3. true/false
- the vector $[1, 1, 1]^T$ is an eigenvector of A. true/false

K5. Let $A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & 7 \\ 0 & 0 & -2 \end{bmatrix}$.

(2) (a) The eigenvalues of A^{-1} are

$1 (1\times) \quad -\frac{1}{2} (2\times)$

(2) (b) Give an eigenvector of A^{-1} corresponding to a negative eigenvalue.

$\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

($1\frac{1}{2}$) **K6.** Circle for each of the following statements about the matrices

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

whether it is true or false:

(a) A is an orthogonal matrix. True/False

(b) B is an orthogonal matrix. True/False

(2) **K7.** Give all possible values for the determinant of an orthogonal matrix.

± 1

(2) **K8.** How can the matrix of the orthogonal projection on Col(A) be found if $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 8 \\ 3 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

short explanation

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad P = B(B^TB)^{-1}B^T$

(2) **K9.** For $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ and $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ it is given that $\mathbf{p}$ is the orthogonal projection of $\mathbf{b}$ on W. Find a en b.

$a = -1$ $\quad b = 9$

K10. Let $A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & b & a \\ 4 & 8 & 9 \end{bmatrix}$ with $a, b \in \mathbb{R}$.

(2) (a) Find all values of a en b that make this matrix symmetric and positive definite.

$a = 8$ $\quad b > 18$

(2) (b) Take $a = 8$ and $b = 18$. Why does there not exist an invertible 3×3-matrix W such that $A = W^TW$?

short explanation

If W exists, A is positive definite, and it isn't with $a=8$ and $b=18$.

END OF SHORT ANSWER QUESTIONS

Naam:

Studienummer:

---

1. Let

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Furthermore $A = PDP^{-1}$. Answer the next questions without computing A! Explain your answers!

(2) (a) Find $\det(A)$.

(2) (b) True or false: $A^{-1} = A$?

(3) (c) Let $\mathbf{x}_0 = (0, 0, 3)^T$. For $k \in \mathbb{N}$ find $A^k \mathbf{x}_0$.

(2) (d) Let

$$R = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Are A and RCR^{-1} equal?

(2) (e) Is A symmetric?

(2) (f) Does there exist a matrix B such that $A = B^T B$?

(2) (g) Find $\text{Nul}(A - 3I)$.

answer to (a)

Method 1: $\det A = \det(PDP^{-1}) = \det P \det D \frac{1}{\det P} = \det D = -1$

Method 2: $\det A =$ product eigenvalues $= -1 \cdot 1 \cdot 1 = -1$

answer to (b)

Note: $D^2 = I$ so $D^{-1} = D$.

So $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1} = PDP^{-1} = A$

True

answer to (c)

Note that -1 and 1 are the eigenvalues of A and that

$E_1 = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\underline{v}_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_2} \right\}$ and $E_{-1} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\underline{v}_3} \right\}$

Now $\underline{x}_0 = 1\underline{v}_1 + 2\underline{v}_2 + \underline{v}_3$ (inspection or solving $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$). So $A^k \underline{x}_0 = 1 \cdot 1^k \underline{v}_1 + 2 \cdot 1^k \underline{v}_2 + (-1)^k \underline{v}_3$

$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + (-1)^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

upturn your sheet!

answer to (d)

If it were true, $\begin{bmatrix}1\\-2\\2\end{bmatrix}$ should be an eigenvector of A with eigenvalue 1. So $\begin{bmatrix}1\\-2\\2\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

Now: $\left[\begin{array}{cc|c}1&-1&1\\-1&0&-2\\0&1&2\end{array}\right] \sim \left[\begin{array}{cc|c}1&-1&1\\0&-1&-1\\0&1&2\end{array}\right] \sim \left[\begin{array}{cc|c}1&-1&1\\0&1&1\\0&0&1\end{array}\right]$ no solutions

So $\begin{bmatrix}1\\-2\\2\end{bmatrix} \notin \text{Span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ 1 so $A \neq RCR^{-1}$

answer to (e)

Note that $\underline{v}_3 \perp \underline{v}_1$ $\frac{1}{2}$ $\underline{v}_3 \perp \underline{v}_2$. By applying GS we can make an orthogonal $\frac{1}{2}$ basis for E_1 which leads to an orthogonal set of 3 eigenvectors. That means that A is orthogonally diagonalizable, so A is symmetric!

answer to (f)

If it exists we have $\underline{x}^T B^T B \underline{x} = \|B\underline{x}\|^2 \geq 0$

So $A = B^T B$ is pos $\frac{1}{2}$ semidefinite, so all eigenvalues of A are ≥ 0. That's not true, so B doesn't exist.

answer to (g)

$Nul(A-3I) = \{\underline{0}\}$ Because if $\underline{x} \neq \underline{0} \in Nul(A-3I)$ we have $(A-3I)\underline{x} = \underline{0}$ $\frac{1}{2}$ so $A\underline{x} = 3\underline{x}$. That means that $\underline{x}$ is evector of A $\frac{1}{2}$ with eva $= 3$. ↯

And surely $\underline{0} \in Nul(A-3I)$ since it is a subspace. $\frac{1}{2}$

Not enough space? Ask for an extra sheet!

END OF OPEN QUESTION 1

Naam: Joost

Studienummer:

---

2. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 1 & 4 & 3 & 4 & 3 \\ 3 & 4 & 1 & 0 & -3 \\ 1 & -1 & -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

and let W be the nullspace of A. Furthermore let

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

(2) (a) Show that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W. Do this without solving a system!!

(3) (b) Find the orthogonal projection $\mathbf{p}$ of $\mathbf{b}$ on W and find the distance of $\mathbf{b}$ to W.

(2) (c) Give a vector $\mathbf{c}$, not equal to $\mathbf{b}$ or $\mathbf{p}$, such that its orthogonal projection on W equals $\mathbf{p}$.

answer to (a)

• Note that $\underline{w}_1$ and $\underline{w}_2$ are independent because none of the two is a saclar mutiple of the other.

• Note that dim Nul A = # colums without a pivot = 2 so the number of vectors in $\{\underline{w}_1, \underline{w}_2\}$ equals dim W

• Note that $A\underline{w}_1 = \underline{0}$ and $A\underline{w}_2 = \underline{0}$, so $\underline{w}_1, \underline{w}_2 \in W$.

We can conclude that $\{\underline{w}_1, \underline{w}_2\}$ is a basis for W.

upturn your sheet!

SEE NEXT PAGE

answer to (b)

First create an orthogonal basis $\{\underline{v}_1, \underline{v}_2\}$ for W by applying Gram-Schmidt: $\underline{v}_1 = \underline{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$v_2 = \underline{w}_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} \underline{v}_1 = \underline{w}_2 - \frac{1}{3}\underline{v}_1 = \frac{1}{3}(3\underline{w}_2 - \underline{v}_1) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \\ 3 \end{bmatrix}$$

take $\underline{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \\ 3 \end{bmatrix}$

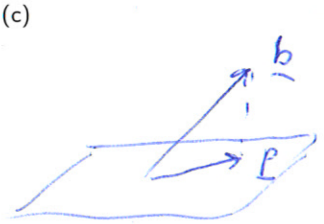
Now the orthogonal projection of $\underline{b}$ on W equals

$$\underline{p} = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} \underline{v}_2 = \frac{2}{3}\underline{v}_1 + \frac{16}{24}\underline{v}_2 = \frac{2}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$$

$$= \frac{2}{3}(\underline{v}_1 + \underline{v}_2) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 2 \end{bmatrix}$$

Check: $\underline{b} - \underline{p} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} \perp \underline{v}_1$ and $\perp \underline{v}_2$. So seems ok.

answer to (c)



Every vector on the line through $\underline{b}$ and $\underline{p}$, so every vector of the form $\underline{v} = \underline{p} + \lambda(\underline{b} - \underline{p})$

Indeed: then $\underline{v} - \underline{p} = \lambda(\underline{b} - \underline{p}) \perp W$ and $\underline{p} \in W$ $\}$ so $\underline{p} = \mathrm{proj}_W(\underline{v})$

Not enough space? Ask for an extra sheet!

END OF OPEN QUESTION 2

TW1030, Tuesday 30 October 2018, open question 3 of 3

Naam: Joost

Studienummer:

3. In $\mathbb{R}^n$ (with $n \geq 2$) are taken two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 1$. With these vectors we define the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\mathbf{x}) = \mathbf{x} - (\mathbf{u} \cdot \mathbf{x})\mathbf{v}$$

(you don't need to prove that T is linear).

(1) (a) Is $\mathbf{v}$ een eigenvector of T? If yes, also give the corresponding eigenvalue.
[REMARK: λ is an eigenvalue of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ if an eigenvector $\mathbf{x} \neq \mathbf{0}$ exists, such that $T(\mathbf{x}) = \lambda\mathbf{x}$.]

(1) (b) (*i*) Show that the nonzero vectors orthogonal with $\mathbf{u}$ are eigenvectors of T. Also give the corresponding eigenvalue.

(2) (*ii*) Why is the dimension of the corresponding eigenspace equal to $n - 1$?

(2) (c) Show, by using (a) and (b), that T is diagonalizable.
[REMARK: a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable if and only if $\mathbb{R}^n$ has a basis containing only eigenvectors of T.]

answer to (a)

Note that $T(\underline{v}) = \underline{v} - (\underline{u} \cdot \underline{v})\underline{v} = \underline{v} - 1\underline{v} = \underline{0}$

Not that $\underline{v} \neq \underline{0}$ (otherwise $\underline{u} \cdot \underline{v} = 0$)

So $\underline{v}$ is an eigenvector of T with eigenvalue 0

answer to (b)

i) Let $\underline{x} \in \mathbb{R}^n$, $\underline{x} \perp \underline{u}$. So $\underline{u} \cdot \underline{x} = \underline{0}$

Then $T(\underline{x}) = \underline{x} - (\underline{u} \cdot \underline{x})\underline{v} = \underline{x} - 0\underline{v} = \underline{x}$

So if $\underline{x} \neq \underline{0}$ then $\underline{x}$ is an eigenvector of T with eigen-value 1.

ii) Note that $E_1 \supset \text{Span}\{\underline{u}\}^\perp$

Since $\text{span}\{\underline{u}\}$ is 1-dimensional, $\text{span}\{\underline{u}\}^\perp$ is $n-1$-dimensional. So $\dim E_1 = n-1$ ½ or $\dim E_1 = n$. The latter is impossible because then $E_1 = \mathbb{R}^n$ and in that situation

upturn your sheet!

answer to (b) continued

every vector of $\mathbb{R}^n$ would be an eigenvector of T with eigenvalua 1. Contradiction with (a).

answer to (c)

We have found two eigenvalues: 0 and 1.

$\dim E_1 = n-1$, so we can find $n-1$ independent eigen-vectors with eigenvalue 1.

$\underline{v}$ is an eigenvector with eigenvalue 0.

This delivers n ~~eig~~ independent eigenvectors.

The form an eigenbasis for $\mathbb{R}^n$ wrt T.

So T is diagonalizable.

Not enough space? Ask for an extra sheet!