Technische Universiteit Delft
Faculteit Elektrotechniek, Wiskunde en Informatica
Mekelweg 4, Delft

Midterm exam Kaleidoscope, (TW1021 and TW1021A)
5 October 2018, 9:00 - 11:00.

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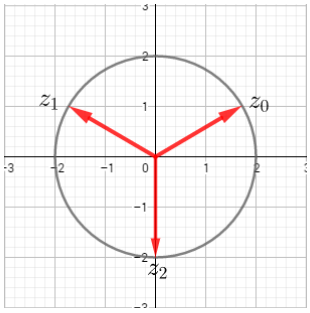
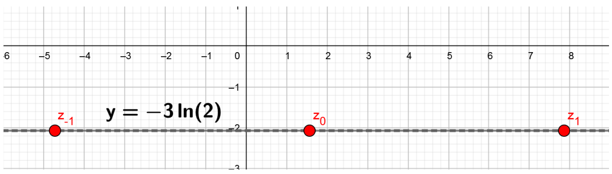
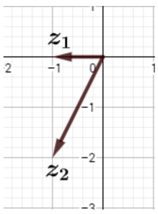
(5) **1.** (a) $z^3 = 8i \Leftrightarrow |z^3| = |8i|$ and $\arg z^3 = \arg 8i$
$|z^3| = |8i| \Leftrightarrow |z|^3 = 8 \Leftrightarrow |z| = 2$ and
$\arg z^3 = \arg 8i \Leftrightarrow 3 \arg z = \frac{\pi}{2} + 2\pi k$ with $k \in \mathbb{Z} \Leftrightarrow \arg z = \frac{\pi}{6} + \frac{2\pi k}{3}$ with $k \in \mathbb{Z}$. So the three solutions are:
$z_0 = 2(\cos\frac{\pi}{6} + \sin\frac{\pi}{6} i) = \sqrt{3} + i$, $z_1 = 2(\cos\frac{5\pi}{6} + \sin\frac{5\pi}{6} i) = -\sqrt{3} + i$ and $z_2 = 2(\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2} i) = -2i$



(4) (b) Let $z = x + yi$ with $x, y \in \mathbb{R}$
Then $\mathrm{e}^{iz} = \mathrm{e}^{i(x+iy)} = \mathrm{e}^{-y+ix}$ and $8i = 8\mathrm{e}^{(\frac{\pi}{2}+2\pi k)i} = \mathrm{e}^{\ln 8 + (\frac{\pi}{2}+2\pi k)i} = \mathrm{e}^{3\ln 2 + (\frac{\pi}{2}+2\pi k)i}$ with $k \in \mathbb{Z}$.
So $y = -3\ln 2$ and $x = \frac{\pi}{2} + 2\pi k$ with $k \in \mathbb{Z}$.
$z_k = \frac{\pi}{2} + 2\pi k - (3\ln 2)i$ with $k \in \mathbb{Z}$ are now the solutions of the equation.

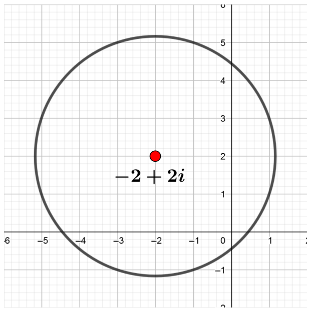


(4) (c) The equation can be written as $(z + (1+i))^2 - (1+i)^2 + 1 + 2i = 0$.
And $(z + (1+i))^2 - (1+i)^2 + 1 + 2i = 0 \Leftrightarrow (z + (1+i))^2 + 1 = 0 \Leftrightarrow (z + (1+i))^2 = -1 \Leftrightarrow$ $z + (1+i) = i \vee z + (1+i) = -i \Leftrightarrow z = -1 \vee z = -1 - 2i$.
So $z_1 = -1$ and $z_2 = -1 - 2i$ are the solutions of the quadratic equation.



(3) **2.** (a) $|z + 2i| = \sqrt{2}|z + 1| \Leftrightarrow x^2 + (y+2)^2 = 2((x+1)^2 + y^2) \Leftrightarrow x^2 + y^2 + 4y + 4 = 2x^2 + 4x + 2 + 2y^2 \Leftrightarrow$ $x^2 + 4x + y^2 - 4y - 2 = 0 \Leftrightarrow (x+2)^2 - 4 + (y-2)^2 - 4 - 2 = 0 \Leftrightarrow (x+2)^2 + (y-2)^2 = 10$.
So $\mathcal{C}$ is the circle with midpoint $-2 + 2i$ and radius $\sqrt{10}$.

(1) (b) Graph of $\mathcal{C}$.



Mark: (total number of points + 4)/4 rounded up to the nearest half.

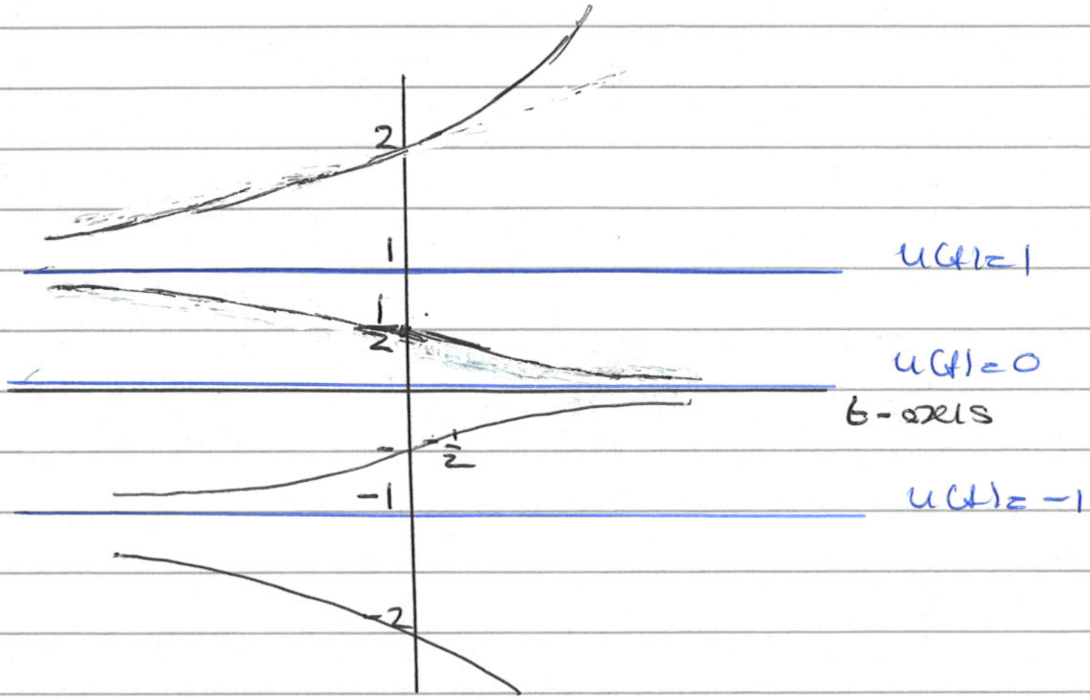
Exam Differential Equations.

① (a) The equilibrium solutions of (1a) are the constant solutions of this differential equation.

So $u'(t) = 0 \Leftrightarrow u(t)(u^2(t) - 1) = 0 \Leftrightarrow$

$u(t) = 0 \vee u^2(t) = 1 \Leftrightarrow u(t) = 0 \vee u(t) = \pm 1$

(b)



$u'(t) > 0 \Leftrightarrow u(t)(u^2(t) - 1) > 0 \Leftrightarrow u(t) > 0 \wedge u^2(t) > 1 \vee$
$u(t) < 0 \wedge u^2(t) < 1$

$\Leftrightarrow u(t) > 1 \vee -1 < u(t) < 0$

(c). If $|u(0)| < 1$ then $\lim\limits_{t \to \infty} u(t) = 0$ and

if $|u(0)| > 1$ then $\lim\limits_{t \to \infty} |u(t)| = \infty$.

$u(t) = 0$ is the only stable equilibrium solution

The solutions $u(t) = \pm 1$ are unstable.

(2) (a) $ty' + 2y = 0 \quad \cdots \quad (2c)$
is the homogeneous solution corresponding to (2a).

$y(t) = 0$ is a solution of (2c)

Let now y be another solution of (2c).

Then $ty'(t) + 2y(t) = 0$.

$$\Leftrightarrow ty'(t) = -2y(t).$$

$$\Leftrightarrow \frac{y'(t)}{y(t)} = -\frac{2}{t}.$$

$$\Leftrightarrow \int \frac{y'(t)}{y(t)}\,dt = -2\int \frac{1}{t}\,dt.$$

$$\Leftrightarrow \ln|y(t)| + C_1 = -2\ln|t| + C_2. \quad (C_1, C_2 \in \mathbb{R}).$$

$\nearrow t>0$

$$\Leftrightarrow \ln|y(t)| = -2\ln t + C_3 \quad (C_3 = C_2 - C_1,\ C_3 \in \mathbb{R})$$

$$\Leftrightarrow \ln|y(t)| = \ln t^{-2} + C_3 \quad (C_3 \in \mathbb{R}).$$

$$\Leftrightarrow |y(t)| = \frac{e^{C_3}}{t^2} \quad (C_3 \in \mathbb{R}).$$

$$\Leftrightarrow y(t) = \pm\frac{e^{C_3}}{t^2} \quad (C_3 \in \mathbb{R}).$$

Together with the earlier found solution:

$y(t) = \frac{D}{t^2}$ $(D \in \mathbb{R})$ is the general solution of (2c).

(b) Let $y(t) = \frac{D(t)}{t^2}$ be a solution of (2a) for certain differentiable function D of t.

Then $y'(t) = \frac{D'(t)}{t^2} - \frac{2D(t)}{t^3}$ so

$$t\,y'(t) + 2y(t) = \frac{\sin t}{t}$$

$$\Leftrightarrow \frac{D'(t)}{t} - \cancel{\frac{2D(t)}{t^2}} + \cancel{\frac{2D(t)}{t^2}} = \frac{\sin t}{t}.$$

$$\Leftrightarrow D'(t) = \sin t. \quad \Leftrightarrow \quad D(t) = -\cos t + E \quad (E \in \mathbb{R}).$$

Thus the general solution of (2a) is given by

$$y(t) = \frac{D(t)}{t^2} = \frac{(-\cos t + E)}{t^2} = \frac{-\cos t}{t^2} + \frac{E}{t^2}. \quad (t>0,\ E \in \mathbb{R}).$$

(c) $$y\left(\frac{\pi}{2}\right) = 1 \Leftrightarrow \frac{-\overset{=0}{\cos\frac{\pi}{2}}}{\left(\frac{\pi}{2}\right)^2} + \frac{E}{\left(\frac{\pi}{2}\right)^2} = 1$$

$$\Leftrightarrow E = \frac{\pi^2}{4}.$$

$$y(t) = \frac{-\cos t}{t^2} + \frac{\pi^2}{4t^2} \quad (t>0).$$

(3) (a) The homogeneous solution corresponding to (3) is

$$u'' - 4u' + 4u = 0. \quad \cdots (3a)$$

$u_1(t) = e^{2t}$, $u_1'(t) = 2e^{2t}$ and $u_1''(t) = 4e^{2t}$.

$u_1''(t) - 4u_1'(t) + 4u_1(t) = 4e^{2t} - 4(2e^{2t}) + 4e^{2t} =$

$4e^{2t} - 8e^{2t} + 4e^{2t} = 0 \quad \forall t \in \mathbb{R}$.

So u_1 is a solution of (3a).

$u_2(t) = te^{2t}$, $u_2'(t) = e^{2t} + 2te^{2t}$ and

$u_2''(t) = 2e^{2t} + 2e^{2t} + 4te^{2t} = 4e^{2t} + 4te^{2t}$

$u_2''(t) - 4u_2'(t) + 4u_2(t) = (4e^{2t} + 4te^{2t})$

$-4(e^{2t} + 2te^{2t}) + 4te^{2t} =$

$\cancel{4e^{2t}} + \cancel{4te^{2t}} - \cancel{4e^{2t}} - \cancel{8te^{2t}} + \cancel{4te^{2t}} = 0 \quad \forall t \in \mathbb{R}$

So u_1 and u_2 are solutions of (3a).

(b) Because u_1 and u_2 are solutions of (3a)

the general solution of this equation

is $u_h(t) = c_1 u_1(t) + c_2 u_2(t) = c_1 e^{2t} + c_2 t e^{2t}$ $(c_1, c_2 \in \mathbb{R})$

(c) Let $y_p(t) = A\cos t + B\sin t$ be a solution of (3)

$y_p'(t) = -A\sin t + B\cos t$ and

$y_p''(t) = -A\cos t - B\sin t$.

$y_p''(t) - 4y_p'(t) + 4y_p(t) = 5\sin t \quad \forall t \in \mathbb{R}$.

$\Leftrightarrow (-A\cos t - B\sin t) - 4(-A\sin t + B\cos t) +$

$4(A\cos t + B\sin t) = 5\sin t \quad \forall t \in \mathbb{R}$.

$\Leftrightarrow (-A - 4B + 4A)\cos t + (-B + 4A + 4B)\sin t = 5\sin t$
$\forall t \in \mathbb{R}$

$\Leftrightarrow (3A - 4B)\cos t + (4A + 3B)\sin t = 5\sin t \quad \forall t \in \mathbb{R}$.

Substituting $t = 0$ and $t = \pi/2$ gives:

$3A - 4B = 0$ and $4A + 3B = 5 \Leftrightarrow \frac{16}{3}B + 3B = 5$

$\Leftrightarrow \frac{25}{3}B = 5$

$3A - 4B = 0 \Leftrightarrow A = \frac{4}{3}B$

$\Leftrightarrow B = \frac{3}{5}$

$\left.\begin{array}{l} A = 4/3\, B. \\ B = 3/5, \end{array}\right\} \Rightarrow A = 4/5.$

so $y_p(t) = \frac{4}{5}\cos t + \frac{3}{5}\sin t$.

The general solution of (3) is given by

$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 t e^{2t} + \frac{4}{5}\cos t + \frac{3}{5}\sin t$.