Assignment exam Applied Functional Analysis
January 9, 2020, 10.45 - 12.30

Solutions should be given in full detail. Results from the AFA Course Notes may be quoted without proof.

Grading: $\frac{1}{4} \cdot (10 + (5+5) + 10 + 10)$

All vector spaces are over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

1. For $1 \leqslant p \leqslant \infty$ consider the right shift operator $T_{\mathrm{r}} \in \mathscr{L}(\ell^p)$:

$$T_{\mathrm{r}}(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots)$$

and the left shift operator $T_{\mathrm{l}} \in \mathscr{L}(\ell^p)$:

$$T_{\mathrm{l}}(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots).$$

Find $\sigma(T_{\mathrm{r}})$ and $\sigma(T_{\mathrm{l}})$.

Hint: Start with finding the eigenvalues of T_{l}.

2. Let T be a bounded operator on a Hilbert space H.

(a) Prove that H admits an orthogonal direct sum decomposition

$$H = \mathrm{Ker}\; T^\star \oplus \overline{\mathrm{Ran}\; T}.$$

(b) Show that if T is a contraction (i.e., $\|T\| \leqslant 1$), then for each $x \in H$ we have $Tx = x$ if and only if $T^\star x = x$. Conclude that H admits an orthogonal direct sum decomposition

$$H = \mathrm{Ker}\; (I - T) \oplus \overline{\mathrm{Ran}\; (I - T)}.$$

Hint: If $Tx = x$, show that $T^\star x - x \perp x$ and deduce that $T^\star x = x$.

3. Let $1 \leqslant p \leqslant \infty$ be fixed and define $J : L^p(0,1) \to L^p(0,1)$ by

$$Jf(t) := \int_0^t f(s)\, \mathrm{d}s, \qquad t \in [0,1].$$

Show that J is bounded and that it maps $L^p(0,1)$ into $W^{1,p}(0,1)$ boundedly. What is the weak derivative of Jf?

4. Suppose $(S(t))_{t \geqslant 0}$ is a C_0-semigroup on a Banach space X whose generator A is bounded. Show that $S(t) = e^{tA}$ for all $t \geqslant 0$ and $\lim_{t \downarrow 0} \|S(t) - I\| = 0$.

-- The end --