Delft University of Technology
Faculty EEMCS
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Exam Complex Function Theory (TW2040)
Monday 1 July 2019; 13:30 – 16:30.

1. Complete the following definitions.

(5) a. The *principal part* of a Laurent series is ...

(5) b. A singularity of an analytic function is *essential* if ...

2. Complete the following statements.

(5) a. A singularity a of an analytic function f is *essential* if in the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ of f at a ...

(5) b. The *Minimum Modulus Principle* for analytic functions states ...

3. Consider the sine function $\sin z$ on the vertical strip $S = \{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\}$.

(5) a. Verify that $\sin z$ is injective on S.

(5) b. Determine the image I of S under the map $\sin z$ and sketch the images of the horizontal lines $\operatorname{Im} z = -1$, $\operatorname{Im} z = 0$ and $\operatorname{Im} z = 2$, and the vertical lines $\operatorname{Re} z = -\frac{\pi}{4}$, $\operatorname{Re} z = 0$ and $\operatorname{Re} z = \frac{\pi}{3}$.

Let $\arcsin : I \to S$ be the inverse of $\sin : S \to I$

(5) c. Use the inverse function theorem to show that $\arcsin$ is analytic on I and determine its derivative.

(5) d. Find a formula for $\arcsin z$ in terms of the logarithm and the square root. Recalculate the derivative of $\arcsin z$.

(5) 4. a. Let $\mathbb{E} = \{z : |z| < 1\}$ and let $f : \mathbb{E} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 2\sqrt{|z|}$ for all z. Prove that $f(0) = 0$ and $|f(z)| \leqslant 2|z|$ for all z.

(5) b. Let $g : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|g(z)| \leqslant |z|^{\frac{3}{2}}$ for all z. Prove that $g(z) = 0$ for all z. *Hint*: Show first that $g(z) = \alpha + \beta z$ for some constants α and β.

(15) 5. Evaluate

$$\int_0^{2\pi} \frac{\sin 2t}{5 - 3\cos t}\, dt$$

using contour integration. Give all details.

6. Let $M = \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$; define $f : M \to \mathbb{C}$ by $f(\frac{1}{n}) = (-1)^n/n$, and let $\mathbb{E} = \{z : |z| < 1\}$ be the unit disc.

(5) a. How many analytic functions $h : \mathbb{E} \to \mathbb{C}$ are there such that $h(z) = f(z)$ for all $z \in M \cap \mathbb{E}$? Justify your answer.

(5) b. How many analytic functions $h : \mathbb{E} \setminus \{0\} \to \mathbb{C}$ are there such that $h(z) = f(z)$ for all $z \in M \cap \mathbb{E}$? Justify your answer.

(15) 7. Evaluate the integral

$$\int_0^{\infty} \frac{\ln x}{x^2 + a^2}\, dx$$

where a is a positive real number. Use contour integration and give all details.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END