Midterm Assignments Complex Function Theory, AM2040
May 2020

- You are allowed to use the book, your notes, etc.
- Every answer must be motivated by a calculation, a logical argumentation or a reference to the theory of Chapters 1,2,3 and 6.1. You cannot refer to results of exercises.
- If the answer is a complex number, write it in the form $a + bi$.
- All (sub)questions have equal weight for the grade.

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1. Determine all solutions of the equation $e^{z^2} = 1$.
2. Compute the limit or show that the limit does not exist.
   a) $\lim_{z\to\infty} e^{-(z+i)^2}$
   b) $\lim_{z\to 0} z\,\mathrm{Log}(z-1)$.
3. Let $f(z) = z\,\mathrm{Re}(z)$. Determine all points $z_0 \in \mathbb{C}$ for which the complex derivative $f'(z_0)$ exists.
4. Let f be an analytic function on a region Ω satisfying
   $$f(z) = u(x) + iv(y), \qquad z = x + iy \in \Omega,$$
   where u and v are real functions. Show that $f(z) = az + b$ for certain constants $a \in \mathbb{R}$ and $b \in \mathbb{C}$.
5. Calculate the following integrals.
   a) $\int_{[z_0,z_1,z_2,z_3]} (z-1)^{\frac{1}{2}}\,dz$, where $z_0 = 1+i$, $z_1 = 2+3i$, $z_2 = 4-i$, $z_3 = 1-i$ and we use the principal branch of the power function.
   b) $\int_{C_2(0)} \frac{\sin(z)}{4z^2+1}\,dz$, where $C_2(0)$ is positively oriented.
6. Find all entire functions f with the property $|f'(z)| \geq 1$ for all $z \in \mathbb{C}$.
   Hint: Liouville's theorem.
7. a) Let $m, n \in \mathbb{N}_0$ with $m \geq n$, and $r > 0$. Show that
   $$\frac{1}{2\pi i}\int_{C_r(1)} \frac{z^m}{(z-1)^{n+1}}\,dz = \binom{m}{n},$$
   where $C_r(1)$ has positive orientation.
   b) Use part a to prove that
   $$\binom{m}{n} \leq \frac{m^m n^{-n}}{(m-n)^{m-n}}.$$