Delft University of Technology
Faculty EEMCS
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Midterm-exam Complex Function Theory (TW2040)
Wednesday 22 May 2019; 13:30 – 16:30.

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1. Complete the following definitions.

(5) a. A domain is said to be *elementary* if …

(5) b. A domain D is *star-shaped* if …

2. Complete the following statements.

(5) a. A function $f : D \to \mathbb{C}$ is (*complex*) differentiable at $a \in D$, with $f'(a) = K$ if and only if there is a function $\varphi : D \to \mathbb{C}$ such that $f(z) = f(a) + \varphi(z)(z - a)$ on D and …

(5) b. A function $f : D \to \mathbb{C}$ is *complex* differentiable at $a \in D$ if and only if it is *real* differentiable at a and …

3. In this problem we consider the *principal branch* of $z^{\frac{1}{2}}$ (defined on $\mathbb{C}_-$). We define two functions, f and g, by

$$f(z) = (z^2 - 1)^{\frac{1}{2}} \qquad \text{and} \qquad g(z) = (z + 1)\left(\frac{z - 1}{z + 1}\right)^{\frac{1}{2}}$$

(4) a. Verify that $f(z)^2$ and $g(z)^2$ are equal to $z^2 - 1$.

(4) b. Verify that $f(x) = g(x) = \sqrt{x^2 - 1}$ when x is *real* and larger than 1.

(4) c. Determine $f(x)$ and $g(x)$ for *real* x that are smaller than -1.

(4) d. Determine for both functions the maximal open sets on which they are well defined and analytic.

4. Define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$ and let S be the square $\{z : 0 \leqslant \operatorname{Re} z \leqslant 1 \text{ and } 0 \leqslant \operatorname{Im} z \leqslant 1\}$.

(6) a. Determine and sketch the image of S under f.

(6) b. Determine and sketch the preimage of S under f.

(6) 5. a. Let $u : \mathbb{C} \to \mathbb{R}$ be defined by $u(x, y) = \cos x \cdot \cos y$. Is there an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $u = \operatorname{Re} f$? If yes: determine all possible such functions. If not: why not?

(6) b. Let $u : \mathbb{C} \to \mathbb{R}$ be defined by $u(x, y) = \cos x \cdot \cosh y$. Is there an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $u = \operatorname{Re} f$? If yes: determine all possible such functions. If not: why not?

6. Evaluate the integral

$$\oint_\alpha \frac{\cos \zeta}{\zeta(\zeta - \pi)^3} \, \mathrm{d}\zeta$$

(7) a. When α is given by $\alpha(t) = 2 \exp(it)$ with $0 \leqslant t \leqslant 2\pi$.

(8) b. When α is given by $\alpha(t) = \pi + 2 \exp(it)$ with $0 \leqslant t \leqslant 2\pi$.

(15) 7. Let n be a natural number. Evaluate

$$\oint_\alpha \left(\frac{\zeta + 1}{\zeta}\right)^n \mathrm{d}\zeta$$

where α is the unit circle, so $\alpha(t) = \exp(it)$ with $t \in [0, 2\pi]$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

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1. a. … every analytic function defined on it has a primitive on it. [Definition II.2.8]

   b. … there is a point $z_\bullet \in D$ such that for each point $z \in D$ the whole line segment joining $z_\bullet$ and z is contained in D. [Definition II.2.6]

2. a. … φ is continuous at a and $\varphi(a) = K$. [Remark I.4.2 (b)]

   b. … the Jacobian of f at a represents multiplication by a complex number. [Remark I.5.1], or
   … the real part u of f and the imaginary part v of f satisfy the Cauchy-Riemann equations: $u_x(a) = v_y(a)$ and $u_y(a) = -v_x(a)$. [Theorem I.5.3]

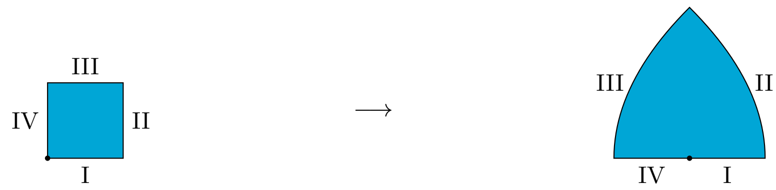
3. a. Since $z^{\frac{1}{2}} \cdot z^{\frac{1}{2}} = z$ for all z, we see at once that $f(z)^2 = z^2 - 1$ and $g(z)^2 = (z+1)^2 \cdot \frac{z-1}{z+1} = z^2 - 1$.

   b. For positive real numbers x we have $x^{\frac{1}{2}} = \sqrt{x}$. If $x \in \mathbb{R}$ and $x > 1$ then $x^2 - 1 > 0$, hence $f(x) = \sqrt{x^2-1}$. Also $g(x) = (z+1)\sqrt{\frac{x-1}{x+1}}$ and because $x - 1 > 0$ and $x + 1 > 0$ we can write $g(x) = (z+1)\frac{\sqrt{x-1}}{\sqrt{x+1}} = \sqrt{x^2-1}$.

   c. If $x \in \mathbb{R}$ and $x < -1$ then $x^2 - 1 > 0$ so, as above, $f(x) = \sqrt{x^2-1}$. For $g(x)$ we must realize that $x - 1 < 0$ and $x + 1 < 0$, so that $\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = \sqrt{\frac{x-1}{x+1}} = \frac{\sqrt{-x+1}}{\sqrt{-x-1}}$ and $g(x) = -(-x-1)\frac{\sqrt{-x+1}}{\sqrt{-x-1}} = -\sqrt{(-x-1)(-x+1)} = -\sqrt{x^2-1}$.

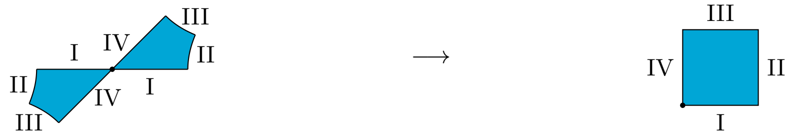
   d. The function $f(z)$ is well-defined and analytic everywhere *except* where $z^2 - 1$ is real and negative, so we must exclude the real interval $[-1, 1]$ and the imaginary axis.
   The function $g(z)$ is well-defined and analytic everywhere *except* where $\frac{z-1}{z+1}$ is real and negative, so we must exclude the real interval $[-1, 1]$ (only).

4. a. Calculate the images of the four sides of the square: I: $(1 - t^2) + 2\mathrm{i}t$; II: $(t^2 - 1) + 2\mathrm{i}t$; III: $-t^2$; IV: t^2; each time $0 \leqslant t \leqslant 1$.



   The angles should be right angles. See also the slides of 2019-05-06.

   b. Now calculate the curves that are mapped onto I, II, III and IV. I: $0 \leqslant x^2 - y^2 \leqslant 1$ and $2xy = 0$, part of the real axis: $-1 \leqslant x \leqslant 1$. II $x^2 - y^2 = 1$ and $0 \leqslant 2xy \leqslant 1$, parts of the hyperbola $x^2 - y^2 = 1$. III: $0 \leqslant x^2 - y^2 \leqslant 1$ and $2xy = 1$, parts of a hyperbola $2xy = 1$. IV: $x^2 - y^2 = 0$ and $0 \leqslant 2xy \leqslant 1$, part of the line $x = y$: $-\frac{1}{2}\sqrt{2} \leqslant x \leqslant \frac{1}{2}\sqrt{2}$.



   The angles should be right angles and the preimage is symmetric with respect to 0. See also the slides of 2019-05-06.

5. a. No: the function u is not harmonic: u_{xx} and u_{yy} are both equal to $-u$, so $u_{xx} + u_{yy} = -2u$ is non-zero.

   b. Yes: the function is harmonic: $u_{xx} = -u$ and $u_{yy} = u$. so $u_{xx} + u_{yy} = 0$. To find all possible functions f we can observe that u is the real part of $\cos z$: use trigonometric formulas to write $\cos(x+\mathrm{i}y) = \cos x \cdot \cos \mathrm{i}y - \sin x \cdot \sin \mathrm{i}y$, then use that $\cos \mathrm{i}y = \cosh y$ and $\sin \mathrm{i}y = i \sinh y$ to get $\cos z = \cos x \cdot \cosh y - i \sin x \cdot \sinh y$. To get all possible functions we must add imaginary constants: $f(z) = \cos z + \mathrm{i}c$ with $c \in \mathbb{R}$.
   Alternatively: use the Cauchy-Riemann equations and find the potential imaginary part by integrating u_x with respect to y and $-u_y$ with respect to x: we get $v(x, y) = -\sin x \cdot \sinh y + h(x)$ or $v(x, y) = -\sin x \cdot \sinh y + k(x)$. The functions h and k must be constant and we get the same anser as above.

See also the next page.

6. a. The function f defined by $f(z) = \cos z/(z-\pi)^3$ is analytic on the star-shaped domain $\{z : \operatorname{Re} z < \pi\}$, so we can apply Cauchy's formula to obtain

$$\oint_\alpha \frac{\cos\zeta}{\zeta(\zeta-\pi)^3}\,\mathrm{d}\zeta = \oint_\alpha \frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta = 2\pi\mathrm{i}\cdot f(0) = \frac{2\pi\mathrm{i}}{(-\pi)^3} = -\frac{2\mathrm{i}}{\pi^2}$$

   b. Now we consider $g(z) = \cos z \cdot z^{-1}$, which is analytic on $\{z : \operatorname{Re} z > 0\}$. Cauchy's formula now gives us

$$\oint_\alpha \frac{\cos\zeta}{\zeta(\zeta-\pi)^3}\,\mathrm{d}\zeta = \oint_\alpha \frac{g(\zeta)}{(z-\pi)^3}\,\mathrm{d}\zeta = \frac{2\pi\mathrm{i}}{2!}g''(\pi) = \pi\mathrm{i}g''(\pi)$$

   But $g''(z) = -\cos z \cdot z^{-1} + 2\sin z \cdot z^{-2} + \cos z \cdot 2z^{-3}$ and so $g''(\pi) = \pi^{-1} - 2\cdot 0\cdot\pi^{-2} - 2\pi^{-3}$. We find that the integral is equal to $\mathrm{i} - \frac{2\mathrm{i}}{\pi^2}$.

7. Method 1: expand $(1+1/z)^n$ using the binomial formula: $\sum_{k=1}^n \binom{n}{k} z^{-k}$; our integral is equal to

$$\sum_{k=1}^n \binom{n}{k} \oint_\alpha \zeta^{-k}\,\mathrm{d}\zeta = \binom{n}{1}\oint_\alpha \frac{1}{\zeta}\,\mathrm{d}\zeta = 2n\pi\mathrm{i}$$

because all integrals, except the one for $k = 1$, are equal to 0.
Method 2: let $f(z) = (z+1)^n$ and note that our integral is equal to

$$\oint_\alpha \frac{f(\zeta)}{\zeta^n}\,\mathrm{d}\zeta = \frac{2\pi\mathrm{i}}{(n-1)!}f^{(n-1)}(0) = 2\pi\mathrm{i}\frac{n!}{(n-1)!} = 2n\pi\mathrm{i}$$

because $f^{(n-1)}(z) = n\cdot(n-1)\cdots 2\cdot(z+1)$.

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