TECHNISCHE UNIVERSITEIT DELFT
Faculteit Elektrotechniek, Wiskunde en Informatica

Exam Discrete Optimisation (WI4227-14)

16 April 2020, 13.30 – 16.30 (3 hours).

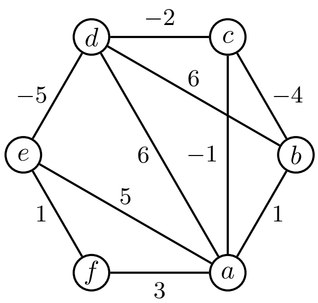
Responsible examinator: dr. D.C. Gijswijt.

This exam consists of **6 problems** worth 90 pts in total.
Pass: 48 points or more, Fail: 47 points or less.
You may use course material (book, notes).
Please write your name and student number on every page that you hand in. **Good luck!**

Name: **Student number:**

Please write the following on the first page of your solutions:
"I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."

[15pts] **1.** In the figure, you see a graph $G = (V, E)$ with edge costs $c : E \to \mathbb{Z}$. Let M be the matroid on E in which $I \subseteq E$ is independent if and only if the subgraph (V, I) has at most one circuit.



[3pts] (a) Find a minimum cost base in this matroid. It suffices to indicate the base in the figure.

[3pts] (b) Let $F = \{e \in E : c(e) > 0\}$. Determine the rank of F.

[3pts] (c) Give a subset $C \subseteq E$ that is a circuit in the matroid M.

[6pts] (d) Consider again the graph G. Call $I \subseteq E$ *good* if $|\delta(v) \cap I| \leq 2$ for every node $v \in V$. Let $\mathcal{I} = \{I \subseteq E : I \text{ is good}\}$.
Is $(V, \mathcal{I})$ a matroid? Prove your answer.

[15pts] **2.** Let $G = (V, E)$ be the complete graph on six nodes. Let $P \subseteq \mathbb{R}^E$ be the TSP polytope and let Q be the polytope given by

$$Q = \{x \in \mathbb{R}^E : x \geq 0, \quad x(\delta(v)) = 2 \quad \text{for all } v \in V\}.$$

[9pts] (a) Give a vector $z \in Q \setminus P$ and a hyperplane separating z from P.

[6pts] (b) Let $x \in \mathbb{R}^E$ be given by

$$x_e = \begin{cases} 0 & \text{if } e \in \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,1\}\} \\ \frac{2}{3} & \text{otherwise} \end{cases}$$

Show that $x \in P$.

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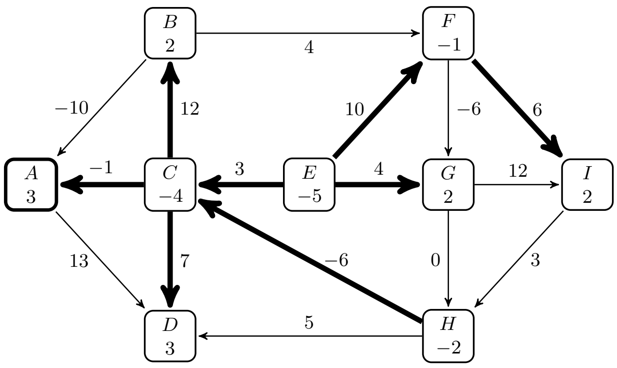
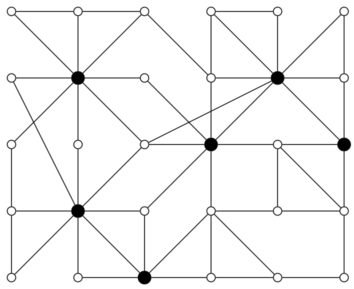
[15pts] **3.** Consider the network in the figure below, which consists of a directed graph, a demand function b on the nodes, and costs c on the edges. There are no upper bounds: $u(a) = \infty$ for every edge a.
Use the network simplex method to solve the minimum cost flow problem, starting from the given tree solution (thick arcs). The vertex A is the root.
In each iteration, give the tree, the associated flow, the vector y (the cost of the paths in the tree from the root to the nodes) and the cost of the flow. You can use the worksheet if you wish.



[15pts] **4.** Consider the graph on 30 nodes below. Give a maximum size matching in this graph[1] and use the Tutte-Berge formula to prove that it is indeed of maximum size.
Hint. You may want to use the set S consisting of the six black nodes.



[15pts] **5.** Let $G = (V, E)$ be an undirected graph. Recall that a *cover* is a subset C of nodes such that every edge has at least one endpoint in C. Let $P \subseteq \mathbb{R}^V$ be the cover polytope:

$$P = \text{conv.hull}(\{\mathbf{1}_C : C \subseteq V \text{ is a cover in } G\}).$$

We also define the polytope Q by

$$Q = \{x \in \mathbb{R}^V : 0 \leq x_v \leq 1 \text{ for all } v \in V, \quad x_u + x_v \geq 1 \text{ for all } uv \in E\}.$$

[9pts] (a) Suppose that G is bipartite. Show that $P = Q$.

[3pts] (b) Suppose that G has an odd circuit through the nodes $v_1, v_2, \ldots, v_{2k+1}$ (in that order). Show that

$$x_{v_1} + x_{v_2} + \cdots + x_{v_{2k+1}} \geq k + 1$$

is a Gomory-Chvátal cutting plane for Q.

[3pts] (c) Suppose that G has five nodes a, b, c, d, e such that any two of these nodes are connected by an edge (i.e. they form a clique). Give a cutting plane proof of

$$x_a + x_b + x_c + x_d + x_e \geq 4$$

from the system describing Q.

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[1] You do not have to follow any particular algorithm, as long as the proof of optimality is correct.

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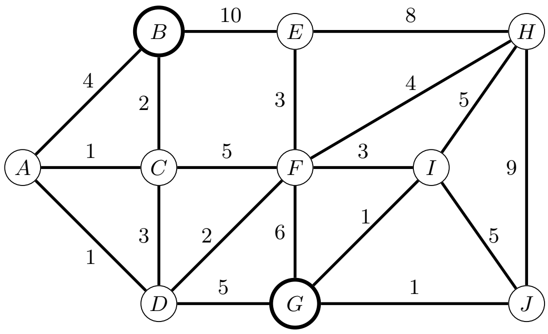
[15pts] **6.** In the figure below, you see a network with costs associated to the edges. The problem is to find a path from B to G of minimum total cost that traverses all edges *at least once* (if an edge is traversed k times, then its cost is counted k times in the cost of the path).

[5pts] (a) Formulate this as a minimum cost T-join problem for some subset T of the nodes. Explain why solving the T-join problem solves the original problem.

[5pts] (b) Explain how this T-join problem can be reduced to a weighted matching problem. Also give the corresponding weight function.

[5pts] (c) Solve the matching problem and give a minimum cost path from B to G traversing every edge of the graph at least once.



End of Exam