Introduction to Mathematical Finance (wi3417tu)
4 november 2014, 18.30–21.30 uur

(No books, no notes.)

Please note: answers should be supplemented by motivation, explanation and/or calculation, whichever may be appropriate. **Point distribution:** each *part* of a question is worth 1 point; the grade equals the number of points earned plus 1.

1. **a.** State the definition of arbitrage.

b. In the binomial tree we assume that $0 < d$; one of the other assumptions is: $d < 1+r$. If $d > 1 + r$ arbitrage is possible; show this by means of a portfolio(computation); use $n = 1$.

2. Consider the following binomial model for the stock price: $S_0 = 20$, $S_1(H) = 40$, $S_1(T) = 10$, $S_2(HH) = 70$, $S_2(HT) = 20$, $S_2(TH) = 30$, $S_2(TT) = 5$. The interest rate $r = 1/4$.

a. Determine the value V_0 of a European put with strike $K = 33$, expiring at $n = 2$.

b. Determine the composition of the replicating portfolio (number of shares and bank balance) at $n = 0$.

3. Consider the binomial model with probabilities $p = 1/3$ and $q = 2/3$. Define $X_i = +1$ if $\omega_i = H$; $X_i = -1$ if $\omega_i = T$. Define $M_0 = I_0 = 0$, $S_0 = 1$, and for $n \geq 1$:

$$M_n = \sum_{j=1}^{n} X_j, \qquad I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j), \qquad S_n = 2^{M_n}.$$

Each of the processes thus defined is *adapted*.

Below you are asked to answer questions on the properties of these processes. In each instance, whether you answer “yes” or “no”, give an argument to support your claim. When, in the course of this, you apply a property of conditional expectations like TOWK, indicate this and explain why it applies.

You may use without proof that $M_0, M_1, \ldots$ is a Markov process.

a. Is $M_0, M_1, \ldots$ a martingale?

b. Is $S_0, S_1, \ldots$ a Markov process? a martingale?

c. Is $I_0, I_1, \ldots$ a Markov process? a martingale? You may use that $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$.

4. Consider the binomial model with parameters: $u = 3/2$, $d = 1/2$, $S_0 = 8$, and $r = -1/4$; yes, a *negative* interest rate!

a. Consider a derivative that pays like a European call with strike $K = 6$ and expiring at $N = 3$, except that this payment is canceled if the stock price process has exceeded 13 on its path.

b. Provide the (general) risk-neutral pricing formula for the time zero price V_0 of a derivative V_N as an expectation under the *real-world* probability $\mathbb{P}$, using the *state price density* ζ. In addition, determine $\zeta(HHH)$ and $\zeta(HTH)$ for the given tree, if $p = q = \frac{1}{2}$.

Short answers:

1a Shreve page 2. This definition is more general than the one on page 13 of Higham.

2a $V_0 = 1028/125 = 8.224$.

2b $V_1(H) = 104/25$, $V_1(T) = 82/5$, whence $\Delta_0 = -51/125 = -0.408$. Bank balance: 16.384.

3a No.

3b Twice: yes. The Markov property can be shown in two ways. The first way: by using that $M_0, M_1, \ldots$ is a Markov process and S_n a function of M_n; demonstrate by using the definition. The second way: by determining $\mathbb{E}_n\left[f(S_{n+1})\right]$ with help of Lemma 2.5.3 and showing that this can be written as $g(S_n)$; it turns out this works with $g(s) := pf(2s)+qf(s/2)$. The quickest way to demonstrate the martingale property goes via the already demonstrated Markov property, choosing $f(x) = x$; evaluating the result, one finds $g(s) = s$, and done!

3c Twice: no; the hardest question. Notwithstanding that $\mathbb{E}_n\left[f(I_{n+1})\right] = g(I_n, M_n)$ for $g(i,m) = pf(i+m)+qf(i-m)$, and furthermore $m^2 = 2i+n$, but this is not sufficient to eliminate M_n (in contrast with Exercise 2.5 from Shreve). The work done also shows that $\mathbb{E}_n\left[I_{n+1}\right] = I_n - \frac{1}{3}M_n$, whence $\mathbb{E}\left[I_{n+1}\right] = \mathbb{E}\left[I_n\right] + n/9$, so no martingale.
Given the right intuition, both no's are easily demonstrated. By drawing the tree 2 or 3 levels deep and determining I_n and M_n in the nodes, one observes that in node H and node T *both* $I_1 = 0$; the value of I_2 is either $+1$ or -1, regardless whether the previous node is H or T, but the probabilities are determined by M_1, whence (for example) $\mathbb{E}_1\left[I_2\right]$ cannot be a function of $I_1 (= 0!)$.
Also, insight in the failure of the martingale property can help you to refute the Markov property.

4a $V_0 = 2/3$.

4b Formula: see Shreve page 70. $\zeta(HHH) = 8/27$ and $\zeta(HTH) = 8/9$.