Exam Intro Math Finance
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All questions have equal weight

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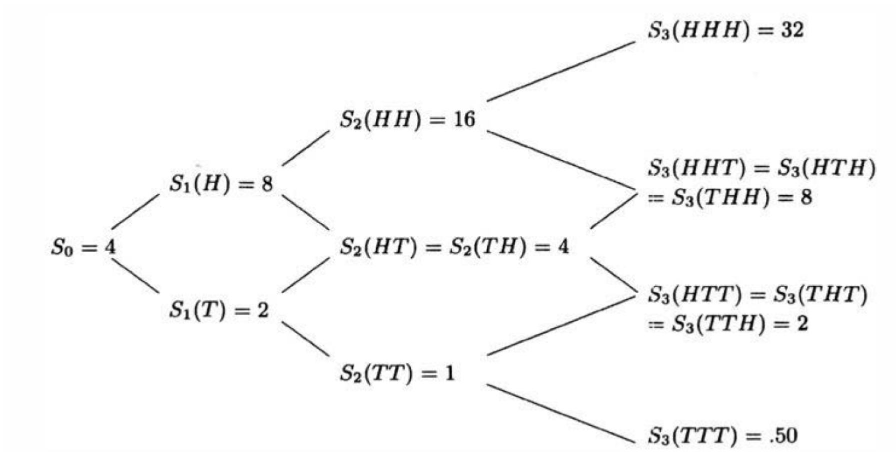
1 Consider the standard two-period model below with interest rate $r = \frac{1}{4}$.



A Compute the strike price K such that the value of a European call C_0 is equal to the value of an European put P_0, if both expire at $N = 2$.

B Let C_n be the value of the call in exercise A, for $n = 0, 1, 2, 3$, and P_n let be the put value of the put. Then $C_n - P_n$ represents one call long and one put short and $C_0 - P_0$ is zero. Compute the positions in the stock Δ_n for $n = 0$ and $n = 1$ in a portfolio that replicates $C_n - P_n$. Explain your result.

C Suppose that the options in exercise A are American instead of European. What can you say about the strike price K such that the value of an *American* call C_0 is equal to the value of an *American* put P_0, if both expire at time $N = 3$. You do not have to compute this new K. You have to determine whether it is higher or lower or exactly equal to the K that you computed in A.

2 Let M_n be the minimum-to-date process defined by $M_n = \min_{0 \le k \le n} S_k$ in the three-period model of Exercise 1. For instance, $M_2(TH) = 2$ because $S_1(T) = 2$ and $S_0 = S_2(TH) = 4$. You may assume that the probability of H and T is equal to $\frac{1}{2}$ in this binomial model.

A Prove that M_0, M_1, M_2, M_3 is a supermartingale.

B Prove that M_0, M_1, M_2, M_3 is a non-Markov process.

C Find the time-zero price for the path-dependent American derivative security whose intrinsic value at each time $n = 0, 1, 2, 3$ is

$$(4 - M_n)^+$$

This intrinsic value is a put on the minimum stock price between time zero and n.

3 Let M_n be the symmetric random walk. Let τ denote the first time the random walk reaches either level 1 or level -5.

$$\tau = \min\{n;\ M_n = 1 \text{ or } M_n = -5\}$$

If the random walk never reaches these levels, we define τ to be infinity.

A Prove that τ is a stopping time.

B Compute the probability that the process stops at -5.

4 In the Ho-Lee model (see Example 6.4.4) the interest rate at time n is

$$R_n(\omega_1 \ldots \omega_n) = a_n + b_n \cdot \#H(\omega_1 \ldots \omega_n)$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are constants used to calibrate the model (note: $b_0 = 0$) and $\#H$ denotes the number of H. The risk-neutral probabilities are $\widetilde{p} = \widetilde{q} = \frac{1}{2}$. We consider a three-period Ho-Lee model with $a_0 = 0.01, a_1 = 0.03, a_2 = 0.05$ and $b_1 = b_2 = -0.01$.

A Compute the forward price $\text{For}_{0,3}$ for the contract that pays R_2 at time three.

B Compute the future price $\text{Fut}_{0,3}$ for the contract that pays R_2 at time three. Compare the two prices in A and B and explain the difference.

ANSWERS

1A 25/4. But if you took $N = 3$ and computed 125/16 that is also okay.

1B At expiry $N = 2$, the portfolio is $C - P = S - K$ and so (discount!) the portfolio is $S - K/(1 + r)^{2-n}$ and $\Delta = 1$.

1C It is lower, because at the K in exercise A the American Put is more expensive than the European Put. To balance the two, the Call needs to increase in value. That happens if K decreases.

2A $M_{n+1} \leq M_n$ by definition. Therefore $\mathbb{E}_n[M_{n+1}] \leq \mathbb{E}_n[M_n] = M_n$.

2B $\mathbb{E}_2[M_3|M_2](HH) = 4$ and $\mathbb{E}_2[M_3|M_2](HT) = 3$. There is no function g such that $g(M_2)$ matches these conditional expectations, because $M_2(HH) = M_2(HT) = 4$.

2C Exercise at T or HTT and else do not exercise. Value $\frac{4}{5} + \frac{16}{125} = \frac{116}{125}$.

3A Stopping time means $\tau(\omega) = n$ then $\tau(\omega') = n$ if the first n outcomes in ω and ω' are the same. In our case, $\tau(\omega) = n$ implies $M_n(\omega) = 1$ or -5 and all $M_i(\omega)$ are not equal to 1 or -5 for earlier i. Clearly, M_i depends on the first $i < n$ tosses only. So if the first tosses of ω and ω' are the same, then the stopping times are the same.

3B According to the Optional Sampling Theorem, the stopped process is a martingale. It stops either at 1 or -5 and its expected value is zero (why?). Therefore it stops at -5 with probability $\frac{1}{6}$.

4A A boring and long computation $\text{For}_0 = \frac{B_{0,2} - B_{0,3}}{B_{0,2}}$ and $B_{0,2} = 0.966$ and $B_{0,3} = 0.929$ and so $\text{For}_0 = 0.0399$.

4B Also boring but short $\text{Fut}_0 = 0.04$ which is higher than For because this product is positively correlated with interest rate.