Delft University of Technology — van Mourik Broekmanweg 6
Faculty EEMCS, Department DIAM — 2628 XE Delft

Midterm AM2080 — 13:30-15:30 October 2, 2020

Responsible examiner: H.P. Lopuhaä
Exam reviewer: S. Grazzi

The exam consists of 5 questions. Each question is graded separately. The average of the 5 questions determines the final grade. Within each question, all parts contribute equally.

Rules and Regulations

1. You can use the book *An Introduction to Mathematical Statistics* by Bijma, Jonker and van der Vaart, the statistical package R and R-studio, hand-outs/solutions from the Brightspace page of AM2080 (2020/2021), and your own personal lecture notes.

2. Scan your work and put it into one pdf file. It is your responsibility that the work is well readable.

3. Try to have the size not too large (20 Mb can lead to delays). The name of your file should be your netid with study number (example: `netid1234567.pdf`).

4. The exam is available on October 2 at 13:30 and ends at 15:30 (Due Date). You have to upload your scan before 16:00 (End Date).

5. If you are allowed extra time, then the exam ends at 15:50 (Due Date) and you have to upload your scan before 16:20 (End Date). Please send the "extra time form" to `h.p.lopuhaa@tudelft.nl` if you have not already done so.

6. Write you name and study number on every page.

7. Write the following line at the top of page 1 of the paper on which you write your solutions and sign it:

   I promise that I have not used unauthorized help from people or other sources for completing my exam. I created the submitted answers all by myself during the time slot that was allocated for that specific exam part.
   Date:
   Name:
   Signature:

1. For $\theta > 0$, define the distribution function

$$F_\theta(y) = \begin{cases} 1 - \theta^2/y^2 & y \geq \theta \\ 0 & y < \theta \end{cases}$$

   (a) Does the family of distributions $\{F_\theta : \theta > 0\}$ form a location-scale family associated with

$$F(x) = \begin{cases} 1 - 1/x^2 & x \geq 1; \\ 0 & x < 1. \end{cases}$$

   If yes, then specify the location parameter a and the scale parameter b. If no, then explain why.

   (b) Derive the α-quantile of F_Y, where $0 < \alpha < 1$.

2. Let $X_1, \ldots, X_n$ be independent with a Bernoulli distribution with parameter $p \in [0, 1]$. Consider the following estimator for the parameter p:

$$T = \sum_{i=1}^{n} c_i X_i$$

   where $c_1, \ldots, c_n \in \mathbb{R}$ are constants.

   (a) Under what condition on $c_1, \ldots, c_n$ is T an unbiased estimator for p?

   (b) Consider the case $n = 2$, with c_1 and c_2, such that T is unbiased. Determine for which $c_1, c_2 \in \mathbb{R}$, the mean squared error is minimal.

3. Let $X_1, \ldots, X_n$ be independent random variables with marginal probability density

$$\mathrm{P}_\theta(X_i = x) = (x-1)\theta^2(1-\theta)^{x-2}, \qquad x = 2, 3, \ldots,$$

   where $\theta \in (0, 1)$.

   (a) Determine the maximum likelihood estimator.

   (b) As prior distribution we choose

$$\pi(\theta) = 15\theta^2(1-\theta), \qquad \theta \in (0, 1).$$

   Determine the Bayes estimator for θ with respect to this prior.

4. Let $X_1, \ldots . X_n$ be independent random variables with distribution function

$$F_\theta(x) = \begin{cases} 1 - (\theta/x)^2 & x \geq \theta; \\ 0 & x < \theta, \end{cases}$$

   for some unknown parameter $\theta > 0$. We want to test $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$ with test statistic

$$T = X_{(1)} = \min\{X_1, \ldots, X_n\}$$

   at significance level α_0. We reject $H_0 : \theta \leq 1$ for large values of $X_{(1)}$.

(a) Show that

$$\mathrm{P}_\theta(X_{(1)} \geq t) = \begin{cases} (\theta/t)^{2n} & t \geq \theta; \\ 1 & t < \theta, \end{cases}$$

and determine the p-value for an observation $t = 1.1$, when $n = 20$.

(b) Show that $c_{\alpha_0} = \alpha_0^{-1/2n}$.

(c) Give the definition of the power function for the test with critical region given by parts (a)-(b) and determine the power at $\theta = 1.25$, when $n = 5$ and $\alpha_0 = 0.05$.

5. Given is a dataset consisting of 9 observations with sample mean 6 and sample variance 4. The observations are assumed to be realizations of independent random variables with a normal distribution with unknown parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. We test $H_0 : \mu = 5$ against $H_1 : \mu \neq 5$ with test statistic $T = \sqrt{n}(\overline{X} - 5)/S_X$.

(a) Compute the p-value corresponding to the observed value for T and report whether you reject the null hypothesis when we perform the test at significance level 10%.

(b) Suppose the critical region is given by $K_T = (-\infty, -3] \cup [3, \infty)$. Compute the size of the test.