DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST LINEAR ALGEBRA 2 (TW2011)
Friday January 26th 2018, 13:30-16:30

The final grade is calculated by computing the sum of all points (maximum 36), adding 4 extra points and dividing the result by 4.

Assignment 1 **(3 pt.)**

Give a reason why each of the following is not an inner product on the given vector spaces:

(a) $\langle (a,b)|(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$, (1 pt.)

(b) $\langle A|B\rangle = \mathrm{Tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$, (1 pt.)

(c) $\langle f(x)|g(x)\rangle = \int_0^1 f'(t)g(t)dt$ on $\mathbb{R}[t]$. (1 pt.)

Assignment 2 **(7 pt.)**

On $M_{2\times 2}(\mathbb{R})$ with Frobenius inner product $\langle A|B\rangle = \mathrm{Tr}(AB^\top)$, let

$$W = \mathrm{Span}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right).$$

(a) Compute (5 pt.)

$$P_W\left(\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\right).$$

(b) Give a basis for $W^\perp$. (2 pt.)

Assignment 3 **(5 pt.)**

(a) Let $V = \mathbb{R}^n$ with a non-standard inner product and let L be the multiplication by a matrix A. Find the matrix of $L^\dagger$ in terms of the matrix A and the metric matrix $\mathbf{G}$. (2 pt.)

(b) Let $V = \mathbb{R}^3$ and consider the non-standard inner product

$$\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3.$$

Compute the adjoint operator $L^\dagger(\mathbf{x})$ of (3 pt.)

$$L(\mathbf{x}) = \begin{pmatrix}3x_1 + x_2\\ x_2 - x_3\\ 5x_1 + x_3\end{pmatrix}.$$

Hint: You can either use part (a) or compute the adjoint operator $L^\dagger(\mathbf{x})$ directly.

Assignment 4 **(3 pt.)**

Let V be a real inner product space. Show that the set of Hermitian operators on V is a subspace of $\mathcal{L}(V)$.

Assignment 5 **(5 pt.)**

(a) Let $\mathcal{B} = \left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$ be an ordered basis for $V = \mathbb{R}^2$. Give the dual basis $\mathcal{B}^*$ to $\mathcal{B}$. (2 pt.)

(b) Let the linear mapping $L : \mathbb{R}_2[t] \to \mathbb{R}$ be defined by

$$L(\mathbf{p}(t)) = \mathbf{p}(0) + \mathbf{p}'(1) + \mathbf{p}''(2)$$

and let $\mathcal{C}$ and $\mathcal{D}$ be the standard ordered bases for $\mathbb{R}_2[t]$ and $\mathbb{R}$, respectively.
Compute the matrix representation (3 pt.)

$$[L^*]_{\mathcal{C}^*\mathcal{D}^*}$$

of the dual mapping $L^* : \mathbb{R}^* \to \mathbb{R}_2[t]^*$, where $\mathcal{C}^*$ and $\mathcal{D}^*$ are the dual bases.

Hint: It is not necessary to compute the dual bases $\mathcal{C}^*$ and $\mathcal{D}^*$.

Assignment 6 **(4 pt.)**

Consider the quadratic form

$$f(\mathbf{x}) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_2x_3 \quad \text{for} \quad \mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3.$$

(a) What are the minimum and maximum values of $f(\mathbf{x})$ under the constraint that $\mathbf{x}$ is a unit vector, i.e. $\|\mathbf{x}\| = 1$? (2 pt.)

(b) For which unit vectors $\mathbf{x}$ does $f(\mathbf{x})$ attain the minimum and maximum values? (2 pt.)

Assignment 7 **(5 pt.)**

(a) Let A be an orthogonal matrix. Show that $\text{cond}(A) = 1$. (2 pt.)

(b) Let U be an unitary 2×2 matrix. Compute (1 pt.)

$$\left(\begin{array}{cc:cc} \multicolumn{2}{c:}{\multirow{2}{*}{$U^\dagger$}} & 0 & 0 \\ & & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc:cc} \multicolumn{2}{c:}{\multirow{2}{*}{U}} & 0 & 0 \\ & & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) Let $\mathcal{B}$ and $\mathcal{D}$ be orthonormal bases for a finite dimensional complex inner product space V. Show that the change-of-basis matrix $P_{\mathcal{DB}}$ is unitary. (2 pt.)

Assignment 8 **(4 pt.)**

Let A be a real symmetric $n \times n$ matrix with n distinct real eigenvalues that are, without loss of generality, ordered as $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then there exists an orthonormal basis of eigenvectors $\mathcal{B} = (\mathbf{b}_1, \ldots \mathbf{b}_n)$. This you do <u>not</u> have to show!

(a) Show that the matrix $B = A - \lambda_1 |\mathbf{b}_1\rangle\langle\mathbf{b}_1|$ has the same eigenvectors and eigenvalues as A except that the largest eigenvalue λ_1 has been replaced by 0. (2 pt.)

(b) Generalize the power method to compute *all* eigenvalue/eigenvector pairs of A. (2 pt.)