DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

**EXAM LINEAR ALGEBRA 2 (AM2010)**
**Tuesday October 27th 2020, 13:30-16:30**

1. This exam is under open-book conditions. This means you are **allowed to use**
   - The book "Applied Linear Algebra" by Lorenzo Sadun;
   - Slides/assignments and the extra notes from Brightspace;
   - Your own personal lecture notes.

   You are **not allowed to use**
   - Help from fellow students;
   - Help from other people;
   - Help from the internet.
2. The exam **starts at 13:30** and **ends at 16:30**. You have **3 hours for completing the exam** and another **30 minutes to scan your work** properly and combine it into **one pdf file**. The scans have to be **clearly readable**! If you are eligible for extra time (Osiris registration is binding) you get 30 minutes extra time.
3. Save the **one pdf file under your name** (example: Jan van Capelle, filename: Capelle.pdf) and upload it to the assignment environment **before 17:00** (regular time) and **17:30** (if you are eligible for extra time), respectively.
4. The upload will **close at 17:45**. If you hand in late, please send an email to `m.moller@tudelft.nl` stating the reason (e.g., technical problems encountered during upload).
5. Try to **keep the size of your pdf file small** (20 Mb can lead to delays).
6. Write your **name and student number on every page** of your exam.
7. Please write the following **declaration at the beginning** of the exam and sign it:

   > "I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud."
8. If you have questions during the exam (imaging when you would raise your hand on a regular on-campus exam) you can send an email to `m.moller@tudelft.nl`.
9. There are **randomly selected face-to-face remote checks between 8:00 and 10:00 on Wednesday October 28th 2020** . If you are selected for a remote check, you will be **notified by email around 17:45 on Tuesday October 27th 2020**. Please be available in the VirtualClassroom at the time stated in the notification email. You will need a webcam and a good, stable internet connection for this. You also need to be able to show your student card and identity card.

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**The final grade is calculated by computing the sum of all points (maximum 36), adding 4 extra points and dividing the result by 4.**

**Assignment 1** **(4 pt.)**
Compute the singular value decomposition of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

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**Assignment 2** **(6 pt.)**
On $M_{2\times 2}(\mathbb{R})$ with Frobenius inner product $\langle A|B\rangle := \mathrm{Tr}(A^\top B)$, let

$$W = \mathrm{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

(a) Compute (3 pt.)

$$P_W\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right).$$

(b) Give a basis for $W^\perp$ and compute (3 pt.)

$$P_{W^\perp}\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right).$$

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**Assignment 3** **(5 pt.)**

(a) Let $V = \mathbb{R}^n$ with a non-standard inner product and let $L$ be the multiplication by a matrix $A$. Find the matrix of $L^\dagger$ in terms of the matrix $A$ and the metric matrix $\mathbf{G}$. (2 pt.)

(b) Let $V = \mathbb{C}^3$ and consider the non-standard inner product

$$\langle \mathbf{x}|\mathbf{y}\rangle = 6x_1y_1 - 2x_2y_2 + 4x_3y_3.$$

Compute the adjoint operator $L^\dagger(\mathbf{x})$ of

$$L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 - 3x_2 - 2x_3 \\ 6x_1 + 2x_2 + 4x_3 \end{pmatrix}.$$

Hint: You can either use part (a) or compute the adjoint operator $L^\dagger(\mathbf{x})$ directly. (3 pt.)

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**Assignment 4** **(3 pt.)**
Let $A \in M_{m,n}(\mathbb{C})$ and define the vector-induced norm $\|A\| := \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$.
Show that $\|A\| = \sigma_{\max}$, where $\sigma_{\max}$ is the largest singular value of $A$.

**Assignment 5** **(4 pt.)**

Consider the quadratic form

$$f(\mathbf{x}) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 8x_2x_3 \quad \text{for} \quad \mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3.$$

(a) Compute the minimum and maximum values of $f(\mathbf{x})$ under the constraint that $\mathbf{x}$ is a unit vector, i.e. $\|\mathbf{x}\| = 1$? (2 pt.)

(b) Compute the vectors $\mathbf{x}$ (with $\|\mathbf{x}\| = 1$), where $f(\mathbf{x})$ attains its the minimum and maximum values? (2 pt.)

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**Assignment 6** **(5 pt.)**

(a) Let $A \in M_{n,n}(\mathbb{R})$ be a real symmetric matrix and $B \in M_{n,n}(\mathbb{R})$ be a real anti-symmetric matrix. Show that $\mathrm{Tr}(AB) = 0$. (3 pt.)

(b) Let $C$ be an orthogonal matrix. Show that $\mathrm{cond}(C) = 1$. (2 pt.)

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**Assignment 7** **(2 pt.)**

Let $A$ be a real symmetric $n \times n$ matrix with $n$ distinct real eigenvalues that are, without loss of generality, ordered as $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then there exists an orthonormal basis of eigenvectors $\mathcal{B} = (\mathbf{b}_1, \dots \mathbf{b}_n)$. This you do <u>not</u> have to show!

Show that the matrix $B = A - \lambda_1 |\mathbf{b}_1\rangle\langle\mathbf{b}_1|$ has the same eigenvectors and eigenvalues as $A$ except that the largest eigenvalue $\lambda_1$ has been replaced by 0.

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**Assignment 8** **(3 pt.)**

Let $A \in M_{n,n}(\mathbb{R})$ and define the sum of the absolute values of the entries of row $i$ of $A$ as

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \quad \text{for } i = 1, 2, \dots, n.$$

Show that for any eigenvalue $\lambda$ of matrix $A$ it holds that

$$|\lambda| \leq \max\{\rho_i(A) : 1 \leq i \leq n\}.$$

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**Assignment 9** **(4 pt.)**

A matrix $B \in M_{n,n}(\mathbb{C})$ is said to be *unitarily equivalent* to $A \in M_{n,n}(\mathbb{C})$ if there exists a unitary matrix $U \in M_{n,n}(\mathbb{C})$ such that $B = U^\dagger A U$.

Show that every matrix $A \in M_{n,n}(\mathbb{C})$ is unitarily equivalent to a triangular matrix.

<u>Hint:</u> This is obvious for $1 \times 1$ matrices. Use this as induction hypothesis ($A \in M_{n,n}(\mathbb{C})$) and prove the statement for $A \in M_{n+1,n+1}(\mathbb{C})$ by induction over $n$. You can use that the product of two unitary matrices $U, Q \in M_{n,n}(\mathbb{C})$ is a unitary matrix without having to prove it.