Assignment 1

(a)(i) Need to find $\underline{0} \in V$ such that $(a,b) \oplus \underline{0} = (a,b)$.

Let $\underline{0} = (x,y)$ then

$(a,b) \oplus (x,y) = (ay + bx, by) = (a,b)$

We want $b = by$, hence, $y = 1$.

We also want $a \cdot 1 + bx = a$, hence, $x = 0$.

We assume that $\underline{\underline{0} = (0,1)}$. 1p

$\underline{0} \in V$ since $0, 1 \in \mathbb{R}$ and $1 > 0$. 1/2p

(a)(ii) Need to find $-(a,b) \in V$ such that $(a,b) \oplus -(a,b) = \underline{0}$

Let $-(a,b) = (x,y)$ then

$(a,b) \oplus (x,y) = (0,1)$

$\Leftrightarrow (ay + bx, by) = (0,1)$

We want $by = 1$ and $ay + bx = 0$.

Hence, $y = (1/b)$ and $x = -(a/b^2)$

We assume that $\underline{-(a,b) = (-a/b^2, 1/b)}$. 1p

$-(a,b) \in V$ since $a, b \in \mathbb{R}$ it follows that

$-a/b^2, 1/b \in \mathbb{R}$ when $b > 0$ and $1/b > 0$ 1/2p

(b) Let $(a,b), (c,d) \in V$ that is $a,b,c,d \in \mathbb{R}$ and $b, d > 0$

Then $(a,b) \oplus (c,d) = (\underbrace{ad + bc}_{\in \mathbb{R}}, \underbrace{bd}_{\in \mathbb{R}})$ and $bd > 0$

1p

Let $(a,b) \in V$ and $t \in \mathbb{R}$. Then $t a b^{t-1} \in \mathbb{R}$ (since $b > 0$) and $b^t \in \mathbb{R}$ and $b^t > 0$. 1p

<u>Assignment 2</u>

(a) We know that the eigenvalues are independent of the basis for V. Hence, let $\mathcal{E} = (1, t, t^2)$.

$$[L]_{\mathcal{E}} = \left([L(1)]_{\mathcal{E}} \quad [L(t)]_{\mathcal{E}} \quad [L(t^2)]_{\mathcal{E}} \right) = \begin{pmatrix} a & b & 2c \\ 0 & a & 2b \\ 0 & 0 & a \end{pmatrix}$$ 1p

Since the eigenvalues of a tridiagonal matrix are the diagonal entries, the eigenvalues of L are $\lambda = a$ with $m_a(\lambda) = 3$. 1p

(b) $L : V \to V$ with $V = \mathbb{R}_2[t]$ being a 3-dimensional vector space. According to theorem 3.8 L is an isomorphism if $\operatorname{rank}(L) = 3$.
From the dimension theorem we have

$$\dim V = \dim(\ker L) + \operatorname{rank}(L)$$

The <u>choice $a = b = c = 1$</u> yields 1p

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{if and only if } x_1 = x_2 = x_3 = 0$$

Hence, $\ker L = \{0\}$ with $\dim(\ker L) = 0$ and $\operatorname{rank}(L) = 3$ argument 1p

(c) Filling in $a = 0$, $b = 1$ and $c = -2$ into the matrix from (a) yields

$$[L]_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ 1p

We also know that $[L]_B = P_{BE} [L]_{\mathcal{E}} P_{EB}$ from theorem 3.2.

$$P_{BE} = P_{EB}^{-1}$$

$$P_{EB} = \left([b_1]_E \quad [b_2]_E \quad [b_3]_E \right)$$

$$= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$P_{BE} = \frac{1}{6}\begin{pmatrix} 4 & 1 & -2 \\ 2 & 5 & -4 \\ -2 & -2 & 4 \end{pmatrix}$$ 1p

$$[L]_B = \frac{1}{3}\begin{pmatrix} -7 & -3 & -17 \\ 1 & 3 & 5 \\ 2 & 0 & 4 \end{pmatrix}$$ 1p

Assignment 3

Let $A^m = O_{n,n}$ and let (λ, v) be an eigenpair of A, that is, $Av = \lambda v$.

$$A^m v = A^{m-1} A v = A^{m-1} \lambda v = \lambda^m v$$ 1p

Since eigenvectors are not the zero vector no power $m > 2$ of A can be the zero matrix unless $\lambda = 0$. 1p

Assignment 4

(a) $\mathcal{B}_1 = \left(\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & \cdots & \\ \vdots & 1 & \\ & 0 & \ddots \end{pmatrix}, \cdots, \begin{pmatrix} 0 & & \vdots \\ & \ddots & 0 \; 0 \\ \cdots & 0 & 1 \end{pmatrix}, \right.$

$1p$

$\begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}, \cdots, \begin{pmatrix} \ddots & & \vdots \\ & \ddots & 0 \\ & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$

$\left. \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & & & \\ 1 & & \ddots & & \\ 0 & & & & \\ \vdots & & & & \end{pmatrix}, \cdots, \begin{pmatrix} & & & \vdots \\ & & & 0 \\ & & \ddots & 1 \\ & & & 0 \; 0 \\ \cdots & 0 & 1 & 0 \; 0 \end{pmatrix}, \cdots \right)$

$\mathcal{B}_2 = \left(\begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 0 & \\ 0 & & & \\ -1 & & & \end{pmatrix} \right)$

$1p$

(b) $\dim V = n^2$

$$\dim \mathcal{B}_1 = \underset{\text{diagonal}}{n} + \underset{\text{off-diag}}{\frac{n^2-n}{2}} = \frac{n^2+n}{2}$$

argument $1p$

$$\dim \mathcal{B}_2 = \frac{n^2-n}{2} \qquad (V = \operatorname{Span}(\mathcal{B}_1) \oplus \operatorname{Span}(\mathcal{B}_2))$$

$$\operatorname{rank}(P_1) = \dim \operatorname{Range}(P_1) = \dim \operatorname{Span}(\mathcal{B}_1) = \frac{n^2+n}{2}$$

$1/2p$

$$\operatorname{rank}(P_2) = \frac{n^2-n}{2} \quad 1/2p$$

Assignment 5

Let A and B be diagonalizable and $AB = BA$ (they commute). It follows that A and B are simultaneously diagonalizable, that is, the same eigenvectors diagonalize A and B possibly with different eigenvalues

$$\Lambda = PAP^{-1} \quad \text{and} \quad \Pi = PBP^{-1}$$

argument 1p

Hence,

$$\Lambda \cdot \Pi = PAP^{-1}PBP^{-1} = P\,AB\,P^{-1}$$

So the eigenvalues of AB are $\lambda_i \pi_i$. 1p

Assignment 6

(a) Taylor series of $p(t) \in \mathbb{R}[t]$ around -1

derivation / argument 1p

$$p(t) = a_0 + a_1 (t+1) + a_2 (t+1)^2 + \dots a_n (t+1)^n$$

$p(t)$ is divisible by $(1+t)^3$ if $a_0 = a_1 = a_2 = 0$.
An equivalence class $[p]$ is thus determined by a_0, a_1 and a_2. V/W is 3-dimensional and a possible basis is $([1], [t+1], [(t+1)^2])$.

1p

(b) $t^2 + 2t = 1 \cdot (t+1)^2 + 0 \cdot (t+1) - 1 \cdot 1$.

Two different representatives from the equivalence class $[t^2+2t]$ are, e.g.

$$p(t) = 1 \cdot (t+1)^2 - 1 + (t+1)^3$$ 1p

$$q(t) = 1 \cdot (t+1)^2 - 1 + (t+1)^4$$ 1p

since $p(t) \neq q(t)$ but $[p(t)] = [q(t)] = [t^2+2t]$.

Assignment 7

(a) $(A - 8I) = \begin{pmatrix} 9 & -9 & -11 & 24 \\ 0 & 0 & 8 & 0 \\ 3 & -3 & -1 & 8 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 9 & -9 & -11 & 24 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 9 & -9 & -11 & 24 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$v_3 = 0$, v_2 and v_4 are free variables

(i) let $v_2 = 1$ and $v_4 = 0 \Rightarrow 9v_1 - 9 = 0 \Leftrightarrow v_1 = 1$

(ii) let $v_2 = 0$ and $v_4 = 3 \Rightarrow 9v_1 + 24 \cdot 3 = 0 \Leftrightarrow v_1 = -8$

Two lin. indep. eigenvectors corresponding to $\lambda_1 = 8$ are

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ 1p \quad \text{and} \quad \underline{v}_2 \begin{pmatrix} -8 \\ 0 \\ 0 \\ 3 \end{pmatrix} \ 1p$$

Hence, a basis for $E_{\lambda_1 = 8}$ is $\mathcal{B}_1 = (\underline{v}_1, \underline{v}_2)$

other eigenvectors are, of course, also correct if they are linearly indepentent

(b) $(A-12I) = \begin{pmatrix} 5 & -9 & -11 & 24 \\ 0 & -4 & 8 & 0 \\ 3 & -3 & -5 & 8 \\ 0 & 0 & 4 & -4 \end{pmatrix}$

$$(A-12I)^2 = \begin{pmatrix} -8 & 24 & 24 & -64 \\ 24 & -8 & -72 & 60 \\ 0 & 0 & 0 & 0 \\ 12 & -12 & -36 & 48 \end{pmatrix}$$

Find b_2 such that $(A-12I)^2 b_2 = 0$
and $(A-12I) b_2 \neq 0$

$$(A-12I)^2 b_2 \to \begin{matrix} -1 & 3 & 3 & -8 \\ 0 & 64 & 0 & -128 \\ 0 & 0 & 0 & 0 \\ 0 & 24 & 0 & -48 \end{matrix}$$

$$\to \begin{matrix} -1 & 3 & 3 & -8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{matrix}$$

$$\to \begin{matrix} -1 & 3 & 3 & -8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Hence, x_3 and x_4 are free variables.
Let $x_3 = 1$ and $x_4 = 0$. Then $x_2 = 0$ and $x_1 = 3$.

Alternatively, let $x_3 = 0$ and $x_4 = 1$. Then $x_2 = 2$ and $x_1 = -2$

Hence a basis for $\tilde{E}_{\lambda_2 = 12}$ is

$$\mathcal{B}_2 = \left(\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right)$$

1p 1p same as in (a)

(c) For the Jordan canonical form of A we need

$$b_1 = (A - 12I)b_2 \quad \text{with} \quad b_2 \in \tilde{E}_{\lambda_2 = 12}$$

Let $b_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ then $b_1 = \begin{pmatrix} 4 \\ 8 \\ 4 \\ 4 \end{pmatrix}$ and

$$P = \begin{pmatrix} 1 & -8 & 4 & 3 \\ 1 & 0 & 8 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 3 & 4 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

1p 1p

Alternatively, let

$b_2 = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ then $b_1 = \begin{pmatrix} -4 \\ -8 \\ -4 \\ -8 \end{pmatrix}$ and

$P = \begin{pmatrix} 1 & -8 & -4 & -2 \\ 1 & 0 & -8 & 2 \\ 0 & 0 & -4 & 0 \\ 0 & 3 & -4 & 1 \end{pmatrix}$ and J as above.

Assignment 8

(a) $\det(A-\lambda I) = (1-\lambda)\left((1-\lambda)^2+4\right) \overset{!}{=} 0$

$\underline{\lambda_1 = 1}$ and $\underline{\lambda_{2,3} = 1 \pm 2i}$ 1p

since $(1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 \overset{!}{=} 0$

$\Leftrightarrow \lambda_{2,3} = 1 \pm \sqrt{1-5} = 1 \pm 2i$

Compute an eigenvector for λ_1

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \rightarrow x_2 = x_3 = 0 \text{ and } x_1 \text{ free} \Rightarrow \underline{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

1p

Compute an eigenvector for λ_2

$$\left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & 2 & 0 \\ 0 & -2 & -2i & 0 \end{array}\right) \rightarrow x_1 = 0 \text{ and, e.g., } x_2 = 1 \Rightarrow x_3 = i$$

Since eigenvectors for complex eigenvalues come in conjugate pairs we have

$$\lambda_2 \rightarrow \underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \qquad \lambda_3 \rightarrow \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$ 1p

(b) $B = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1}$

$= P \operatorname{diag}\left(e^1, e^{1+2i}, e^{1-2i}\right) P^{-1}$ 1p

with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

1p

side calculation

$$\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}^{-1} = \frac{1}{-i-i}\begin{pmatrix}-i & -1\\ -i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$$

$$B = P\ \mathrm{diag}\left(e^{1},\ e^{1}e^{2i},\ e^{1}e^{-2i}\right) P^{-1}$$

$$= e^{1}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{e^{2i}+e^{-2i}}{2} & -\frac{e^{2i}-e^{-2i}}{2}i\\ 0 & \frac{e^{2i}-e^{-2i}}{2}i & \frac{e^{2i}+e^{-2i}}{2}\end{pmatrix}$$ 1p