Exam Martingales, Brownian motion and stochastic calculus (WI4430).
Thursday 30th of January, 13:30-16:30.
Rooms: LR/CZ: E-F-G.

a) The exam has a theory part: questions 1 and 2, each on 10 points, and an exercise part (the remaining questions) on 20 points. The exercise part consists of 10 questions each on 2 points.

b) No books, notes, calculators are allowed on the exam.

c) The second reader of the exam is Dr. Ludolf Meester.

---

1. a) State and prove the martingale convergence theorem. If you prove the L^2 version, then prove also the Kolmogorov-Doob inequality (8 points).

   b) Give an explicit example of a martingale that converges almost surely, and also in L^2 (2 points).

2. a) Prove the formula for the quadratic variation of Brownian motion (5 points).

   b) Show that with probability equal to 1, there does not exist a (non-empty) open interval of time such that Brownian motion is monotone on that interval (3 points).

   c) Give the definition of a Gaussian process (2 points).

3. Let $Y_i, i = 1, 2, \ldots$ denote mutually independent random variables taking the values ± 1 with probabilities $\mathbb{P}(Y_i = 1) = p = 1 - \mathbb{P}(Y_i = -1)$. We assume $p > 1/2$. We further denote $\mathscr{F}_n = \sigma\{Y_i, 1 \leq i \leq n\}$ the natural filtration, and $S_n = \sum_{i=1}^n Y_i$.

   a) Compute, for $\lambda \in \mathbb{R}$ and $n \geq 2$ the conditional expectation

   $$\mathbb{E}(e^{\lambda S_n} | \mathscr{F}_{n-1}).$$

   b) Show that for $\lambda = \log((1-p)/p)$, $X_n = e^{\lambda S_n}$ is a martingale.

   c) Show that the martingale from item b) converges almost surely to zero, but not in L^1.

d) Let $a < 0 < b$ denote two integers. Let $\tau = \inf\{n \geq 1 : S_n \in \{a, b\}\}$. By stopping appropriate martingale(s), compute the expectation $\mathbb{E}(\tau)$. In this item you are allowed to use martingale stopping without further justification.

e) Show that

$$M_n := \sum_{i=1}^{n} (Y_i Y_{i-1} - (2p-1) Y_{i-1})$$

is a martingale (here we put $Y_0 = 0$). Does this martingale converge in L^2?

4. We denote by $\{W(t) : t \geq 0\}$ Brownian motion, with associated natural filtration $\mathscr{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$.

a) Compute the conditional expectation

$$\mathbb{E}(W(t)W(2t)W(3t)|\mathscr{F}_t).$$

b) Show that $|(W(t))^2 - t|^3$ is a sub-martingale (w.r.t. the natural filtration).

c) By stopping an appropriate martingale, show that for $a > 0$, the stopping time $\tau_a = \inf\{t > 0 : W(t) = a\}$ satisfies

$$\mathbb{E}(e^{-\lambda \tau_a}) = e^{-\sqrt{2\lambda} a}$$

for all $\lambda \geq 0$. Justify properly the steps where you use martingale stopping.

d) Show that for every $t > 0$, $W(s) = 0$ infinitely often in $[0, t]$ with probability equal to 1. Hint: show that for all $a > 0$, $M_a = \max_{0 \leq s \leq a} W(s)$ is strictly positive with probability one, and by symmetry, $m_a = \min_{0 \leq s \leq a} W(s)$ is strictly negative. Conclude then that $[0, a]$ contains $s > 0$ such that $W(s) = 0$.

e) Show that the process defined by $Z(t) = e^{-t} W(e^{2t})$ is a Gaussian process, and compute its covariance function $c(t, s) = \text{cov}(Z(t), Z(s))$.