Exam Martingales, Brownian Motion, and Stochastic Processes (WI 4430).

12th of april 2016, 13:30-16:30.

No books or notes allowed.

Responsible of the course: Prof. F. Redig

Second reader exam: Dr. Ludolf Meester

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a) The exam consists of two theory questions, each on 10 points, followed by exercises. The exercises consist of 10 small questions each on 2 points.

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1 Theory Questions.

1) State and prove the martingale convergence theorem. If you give the L^2 proof, then prove also Kolmogorov's maximal inequality. If you give the proof based on upcrossings, then prove also the Doob's upcrossing inequality.

2) a) Give the definition of Brownian motion.
   b) Derive the explicit formula for the probability density of the first hitting time of $a > 0$ for Brownian motion.

2 Exercises.

1) Let $X_i, i \in \mathbb{N}$ be independent random variables with

$$\mathbb{P}(X_i = 10) = 3/4 = 1 - \mathbb{P}(X_i = 1/10).$$

Consider

$$Y_n = \prod_{i=1}^{n} X_i$$

We denote by $\mathscr{F}_n$ the filtration generated by $\{X_i, i \leq n\}$.

a) Compute $\mathbb{E}(Y_n|\mathscr{F}_{n-1})$ and conclude whether Y_n is a sub or supermartingale.

b) Show that $Y_n^{1/n}$ converges almost surely to a constant and compute this constant.

c) Determine the constant a such that $\log Y_n - an$ is a martingale.

d) Define the random time

$$\tau = \inf\{n \in \mathbb{N} : Y_n \geq 100\}$$

Show that τ is a stopping time which is finite almost surely.

e) By stopping the martingale of item c), compute a lower bound for the expectation $\mathbb{E}(\tau)$ of the stopping time of item d).

2) The Ornstein Uhlenbeck process $\{x_t : t \geq 0\}$ with parameters $\mu \in \mathbb{R}$ and $\theta > 0$, starting at $x_0 \in \mathbb{R}$ is defined as follows

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + e^{-\theta t} \int_0^t e^{\theta s} dW_s \tag{1}$$

a) Compute the expectation $\mathbb{E}(x_t)$.

b) Compute the covariance $\mathrm{cov}(x_t, x_s)$, for $0 < s < t$.

c) Show that the law of the process $\{X_t, t \geq 0\}$ defined via

$$X_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{e^{-\theta t}}{\sqrt{2\theta}} W_{e^{2\theta t}-1}$$

is the same as that of the Ornstein Uhlenbeck process $\{x_t, t \geq 0\}$ (hint: Use that both processes are Gaussian. Therefore it is sufficient to show that they have equal mean and covariance).

d) Assume now that x_0 in (1) is chosen normally distributed with mean μ and variance $\frac{1}{2\theta}$ (and independent of the Brownian motion $\{W_t : t \geq 0\}$. Show then that at any later time $t > 0$, x_t is normally distributed with mean μ and variance $\frac{1}{2\theta}$.

3) Let $\{W_t : t \geq 0\}$ be a Brownian motion, and $a > 0$. Denote by $\mathscr{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$ the associated filtration. Determine the function $g(s, W_s)$ such that

$$e^{aW_t - \frac{a^2}{2}t} = \int_0^t g(W_s, s) dW_s.$$