Solutions Exam 26-01-2016 (exercises)

1a) Y_n is $\mathcal{F}_n$-measurable and

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = \mathbb{E}(Y_{n-1} \cdot X_n \mid \mathcal{F}_{n-1})$$

$$= Y_{n-1} \cdot \mathbb{E}[X_n \mid \mathcal{F}_{n-1}]$$

$$= Y_{n-1} \cdot \mathbb{E}(X_n) \qquad (\text{because } X_n \perp \mathcal{F}_{n-1})$$

$$= Y_{n-1} \cdot \left(2 \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2}\right) \geq Y_{n-1}$$

so Y_n is a sub-martingale

b) $\mathbb{E}[Y_{n-1} Y_n^2 \mid \mathcal{F}_{n-1})$

$$= \mathbb{E}(Y_{n-1}^3 X_n \mid \mathcal{F}_{n-1})$$

$$= Y_{n-1}^3 \cdot \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = Y_{n-1}^3 \cdot \left(\tfrac{5}{4}\right)$$

c) $\log Y_n^{\frac{1}{n}} = \frac{1}{n} \sum_{i=1}^{n} \log X_i$.

by the strong law of large numbers (SLLN) this converges almost surely to

$$\mathbb{E}(\log X_1) = \tfrac{1}{2} \log 2 + \tfrac{1}{2} \log \tfrac{1}{2} = 0$$

The conditions of SLLN are satisfied because the variables $Y_i := \log X_i$ are independent, identically distributed and have finite variance and finite expectation.

2 a) $E(X_t^2) = E((\mu t + W_t)^2)$

$$= E(\mu^2 t^2 + 2\mu t W_t + W_t^2)$$

$$= \mu^2 t^2 + 2\mu t\, E(W_t) + E(W_t^2)$$

$$= \mu^2 t^2 + 2\mu t \cdot 0 + t$$

$$= \mu^2 t^2 + t$$

$$E(e^{\lambda X_t} X_s^2) = E(e^{\lambda(X_t - X_s)} e^{\lambda X_s} X_s^2)$$

$$= E\left(e^{\lambda(\mu(t-s) + (W_t - W_s))} e^{\lambda \mu s + \lambda W_s} (\mu s + W_s)^2\right)$$

$$= e^{\lambda \mu t} E\left(E(e^{\lambda(W_t - W_s)} / \mathcal{F}_s) \cdot e^{\lambda W_s} (\mu s + W_s)^2\right)$$

$$= e^{\lambda \mu t} e^{\frac{1}{2}\lambda^2(t-s)} E\left(e^{\lambda W_s}(\mu^2 s^2 + 2\mu s W_s + W_s^2)\right)$$

$$= e^{\lambda \mu t} e^{\frac{1}{2}\lambda^2(t-s)} \left(e^{\frac{1}{2}\lambda^2 s} \mu^2 s^2 + 2\mu s\, E(W_s e^{\lambda W_s}) + E(W_s^2 e^{\lambda W_s})\right)$$

we have

$$\mathbb{E}(e^{\lambda W_s}) = e^{\frac{1}{2}\lambda^2 s}$$

hence

$$\mathbb{E}(W_s e^{\lambda W_s}) = \lambda s e^{\frac{1}{2}\lambda^2 s} \quad \left(= \frac{d}{d\lambda}\mathbb{E} e^{\lambda W_s}\right)$$

$$\mathbb{E}(W_s^2 e^{\lambda W_s}) = s e^{\frac{1}{2}\lambda^2 s} + s^2\lambda^2 e^{\frac{1}{2}\lambda^2 s} \quad \left(= \frac{d^2}{d\lambda^2}\mathbb{E}(e^{\lambda W_s})\right)$$

so

$$\mathbb{E}(e^{\lambda X_t} X_s^2)$$

$$= e^{\lambda\mu t} e^{\frac{1}{2}\lambda^2(t-s)} \Big(e^{\frac{1}{2}\lambda^2 s}\mu^2 s^2 + 2\mu s^2 \lambda e^{\frac{1}{2}\lambda^2 s}$$

$$+ s e^{\frac{1}{2}\lambda^2 s} + s^2\lambda^2 e^{\frac{1}{2}\lambda^2 s}\Big)$$

$$= e^{\lambda\mu t} e^{\frac{1}{2}\lambda^2 t}\left(\mu^2 s^2 + 2\mu s^2\lambda + s + s^2\lambda^2\right)$$

b) $X_t - \mu t$ is clearly integrable, and $\mathcal{F}_t$-measurable

moreover $X_t - \mu t = W_t$, hence, for $0 \leq s < t$

$$\mathbb{E}[W_t \mid \mathcal{F}_s] = E(W_s + (W_t - W_s) \mid \mathcal{F}_s)$$

$$= W_s + \mathbb{E}(W_t - W_s \mid \mathcal{F}_s)$$

$$= W_s + \mathbb{E}(W_t - W_s) \qquad (W_t - W_s \perp \mathcal{F}_s)$$

$$= W_s = X_s - \mu s$$

So $X_t - \mu t$ is a martingale

Next, $(X_t - \mu t)^2 - t = W_t^2 - t$ is clearly integrable and $\mathcal{F}_t$-measurable

$$\mathbb{E}[W_t^2 - t \mid \mathcal{F}_s]$$

$$= \mathbb{E}((W_s + W_t - W_s)^2 \mid \mathcal{F}_s) - t$$

$$= \mathbb{E}(W_s^2 + 2(W_t - W_s)W_s + (W_t - W_s)^2 \mid \mathcal{F}_s) - t$$

$$\overset{(1)}{=} W_s^2 + 2W_s \, E(W_t - W_s \mid \mathcal{F}_s) + \mathbb{E}((W_t - W_s)^2 \mid \mathcal{F}_s) - t$$

$$\overset{(2)}{=} W_s^2 + 0 + (t-s) - t$$

$$= W_s^2 - s$$

in (1) we used take-out-what is known, and in (2) $W_t - W_s \perp \mathcal{F}_s$.

so we find

$$\mathbb{E}[\tilde{M}_t \mid \mathcal{F}_s] = \tilde{M}_s$$ and we conclude

that $\tilde{M}_t$ is a martingale.

c) $e^{-2\mu X_t}$ is clearly integrable

and $\mathcal{F}_t$ - measurable

$$\mathbb{E}\left(e^{-2\mu X_t} \mid \mathcal{F}_s \right)$$

$$= e^{-2\mu^2 t}\, \mathbb{E}\left(e^{-2\mu (W_t - W_s)} \mid \mathcal{F}_s \right) e^{-2\mu W_s}$$

$$= e^{-2\mu^2 t}\, e^{2\mu W_s}\, e^{\frac{1}{2} 4\mu^2 (t-s)}$$

$$= e^{-2\mu W_s - 2\mu^2 s} = e^{-2\mu (W_s + \mu s)}$$

$$= e^{-2\mu X_s}$$

hence $\mathbb{E}\left(e^{-2\mu X_t} \mid \mathcal{F}_s \right) = e^{-2\mu X_s}$

so $M_t = e^{-2\mu X_t}$ defines a martingale d)

$$\mathbb{E}\left(e^{-2\mu X_T}\right)$$

$$= P(X_T = a)\, e^{-2\mu a} + P(X_T = -a)\, e^{2\mu a}$$

by the martingale property and the fact that $\mathbb{E}(T < \infty)$

$$\mathbb{E}\left(e^{-2\mu X_T}\right) = \mathbb{E}\left(e^{-2\mu X_0}\right) = 1$$

so we find

$$P(X_T = a)\, e^{-2\mu a} + P(X_T = -a)\, e^{2\mu a} = 1$$

$$P(X_T = a) + P(X_T = -a) = 1$$

from which we solve

$$P(X_T = a) = \frac{e^{2\mu a} - 1}{e^{2\mu a} - e^{-2\mu a}}$$

e) $\mathbb{E}(X_\tau - \mu\tau) = 0$

$$\mathbb{E}(X_\tau) = a\,P(X_\tau = a) + (-a)\,P(X_\tau = -a)$$

so $\mathbb{E}(\tau) = \frac{1}{\mu}\left(a\,P(X_\tau = a) + (-a)\,P(X_\tau = -a)\right)$

$$= \frac{1}{\mu}\left\{ \frac{a(e^{2\mu a} - 1)}{(e^{2\mu a} - e^{-2\mu a})} - \frac{a(1 - e^{-2\mu a})}{e^{2\mu a} - e^{-2\mu a}} \right\}$$

$$= \frac{1}{\mu}\left\{ \frac{a(e^{2\mu a} + e^{-2\mu a} - 2)}{e^{2\mu a} - e^{-2\mu a}} \right\}$$

3) $f(W_t, t) - f(W_0, t) = \int_0^t \frac{\partial f}{\partial x}(W_s, s)\,dW_s + \frac{1}{2}\int \frac{\partial^2 f}{\partial x^2}(W_s, s)\,ds + \int_0^t \frac{\partial f}{\partial t}(W_s, s)\,ds$

applied to $f(x,t) = x^3 - 3tx$

gives

$$W_t^3 - 3tW_t = \int_0^t (3W_s^2 - 3s)\,dW_s$$

$$+ \int_0^t \Big(\frac{1}{2}\Big[\big(\underbrace{6W_s}\big)\Big] - \underbrace{3W_s}\Big)\,ds$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial f}{\partial t} = -3x$$

$$= \int_0^t (3W_s^2 - 3s)\,dW_s$$

4. Itô isometry gives

$$\mathbb{E}\left(\left(W_t^3 - 3tW_t\right)^2 \right)$$

$$= \mathbb{E}\left(\left(\int_0^t 3\left(W_s^2 - s\right) dW_s \right)^2 \right)$$

$$= \mathbb{E}\left(\int_0^t \left(3W_s^2 - 3s\right)^2 ds \right)$$

$$= \int_0^t 9\,\mathbb{E}\left(W_s^4\right) - 18s\,\mathbb{E}W_s^2 + 9s^2 \, ds$$

$$= \int_0^t 9 \cdot 3s^2 - 18s^2 + 9s^2 \, ds$$

$$= 27\frac{t^3}{3} - 18\frac{t^3}{3} + 9\frac{t^3}{3}$$

$$= 9t^3 - 6t^3 + 3t^3 = 6t^3.$$

Solutions Exam 12-04-2016

Exercises

a) $Y_n = Y_{n-1} \cdot X_n$

so

take out what is known

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = Y_{n-1} \, \mathbb{E}[X_n \mid \mathcal{F}_{n-1}]$$

$$= Y_{n-1} \cdot \mathbb{E}(X_n) \quad \text{(independence)}$$

$$= Y_{n-1} \left(10 \cdot \frac{3}{4} + \frac{1}{10} \cdot \frac{1}{4} \right)$$

$$\geq Y_{n-1}$$

because $\frac{30}{4} + \frac{1}{40} \geq 1$

so Y_n is clearly $\mathcal{F}_n$ - measurable and integrable, so

$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] \geq Y_{n-1}$ gives that

Y_n is <u>sub</u> martingale

b) By the law of large numbers

$$\log Y_n = \frac{1}{n} \sum_{i=1}^{n} \log Y_i \longrightarrow \mathbb{E}(\log Y_1)$$

$$= \frac{3}{4} \log 10 + \frac{1}{4} \log \frac{1}{10} = (\log 10) \cdot \frac{1}{2}$$

so

$$Y_n \longrightarrow \sqrt{10} = e^{\frac{1}{2} \log 10}$$

c) $\mathbb{E}(\log Y_n) = \log \sqrt{10} \cdot n$ so

$\log Y_n - \log \sqrt{10}\, n$ is a martingale

because it is of the form

$$M_n = \sum_{i=1}^{n} Z_i \qquad \text{with} \qquad Z_i = \left(\log X_i - \mathbb{E}(\log X_i)\right)$$

being independent, identically distributed random variables with expectation $\mathbb{E}(Z_i) = 0$.

As a consequence

$$\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] = \mathbb{E}(M_{n-1} + Z_n \mid \mathcal{F}_{n-1})$$

$$= M_{n-1} + \underbrace{\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}]}$$ (take out what is known)

$$= M_{n-1} + \mathbb{E}(Z_n)$$ independent

$$= M_{n-1}$$

Moreover, M_n is clearly $\mathcal{F}_n$- measurable (depends only on $X_1, \dots, X_n$)

d) The event $\tau > n$ is $\{ Y_k < 100 \ \forall k \le n \} \in \mathcal{F}_n$ hence $(\tau \le n) \in \mathcal{F}_n$ so τ is stopping time.

τ is finite a.s. because by by we have $Y_n^{1/n} \longrightarrow \sqrt{10}$ so $Y_n \to \infty$ almost surely.

e) $\mathbb{E}(\log Y_{\cancel{n}\,\tau} - a\tau) \geq \log 100 - a\,\mathbb{E}(\tau)$

so $\mathbb{E}(\tau) \geq \dfrac{\log(100)}{\log(\sqrt{10})} = 4$

2 a) Take expectation in (1)

$$\mathbb{E}(x_t) = \mathbb{E}(x_0 e^{-\theta t}) + \mu(1 - e^{-\theta t}) + e^{-\theta t}\mathbb{E}\int_0^t e^{\theta s} dW_s$$

because $\int_0^t e^{\theta s} dW_s$ is a martingale, we have

$$\mathbb{E}\left(\int_0^t e^{\theta s} dW_s\right) = 0$$

hence

$$\mathbb{E}(x_t) = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

b) $\mathrm{Cov}(x_t, x_s)$: we compute first

$$\mathrm{Cov}\left(\int_0^t e^{\theta r} dW_r, \int_0^s e^{\theta r'} dW_{r'}\right)$$

$$= \mathbb{E}\left(\int_0^t e^{\theta r} dW_r \cdot \int_0^s e^{\theta r'} dW_{r'}\right)$$

$$= \mathbb{E}\left(\left(\int_s^t e^{\theta r} dW_r + \int_0^s e^{\theta r} dW_r\right) \cdot \left(\int_0^s e^{\theta r'} dW_{r'}\right)\right)$$

$$= \mathbb{E}\left(\left(\int_0^s e^{\theta r} dW_r\right)^2\right)$$ (because $\int_s^t e^{\theta r} dW_r$ and $\int_0^s e^{\theta r} dW_r$ are independent)

$$= \int_0^s e^{2\theta r} dr$$ (Ito isometry)

$$= \frac{(e^{2\theta s} - 1)}{2\theta}$$

so covariance equals

$$\mathrm{Cov}\left(e^{-\theta t} \int_0^t e^{\theta s'} dW_{s'} ,\ e^{-\theta s} \int_0^t e^{\theta s'} dW_{s'} \right)$$

$$= e^{-\theta(t+s)} \frac{(e^{2\theta s} - 1)}{2\theta}$$

$$= \frac{e^{-\theta(t-s)} - e^{-\theta(t+s)}}{2\theta}$$

c/ Clearly expectations are equal, so it suffices to show that, for $t \geq s$

$$\mathrm{Cov}\left(\frac{e^{-\theta t}}{\sqrt{2\theta}} W_{e^{2\theta t}-1} ,\ \frac{e^{-\theta s}}{\sqrt{2\theta}} W_{e^{2\theta s}-1} \right)$$

$$= \frac{e^{-\theta(t-s)} - 1}{2\theta}$$

To see this notice that

$\mathrm{Cov}(W_t, W_s) = t \wedge s$, so

$$\mathrm{Cov}\left(\frac{e^{-\theta t}}{\sqrt{2\theta}} W_{e^{2\theta t}-1} ,\ \frac{e^{-\theta s}}{\sqrt{2\theta}} W_{e^{2\theta s}-1} \right)$$

$$= \frac{1}{2\theta} e^{-\theta(t+s)} \left(e^{2\theta s} - 1 \right) = \frac{1}{2\theta} \left(e^{-\theta(t-s)} - e^{-\theta(t+s)} \right.$$

d) If $x_0 \overset{d}{=} N(\mu, \frac{1}{2\theta})$

then

$x_0 e^{-\theta t} + \mu(1-e^{-\theta t})$

$$\simeq N(\mu, \frac{1}{2\theta} e^{-2\theta t})$$

and we have

$$\frac{e^{-\theta t}}{\sqrt{2\theta}} W_{e^{2\theta t}-1} \overset{d}{=} N(0, \frac{1}{2\theta} e^{-2\theta t}(e^{2\theta t}-1))$$

$$= N(0, \frac{1}{2\theta}(1-e^{-2\theta t}))$$

So the sum of these two independent normals is

$N(\mu, \frac{1}{2\theta})$, as had to be shown.

3) Use Ito's formula:

define

$$f(t, x) = e^{ax - \frac{a^2}{2}t}$$

then $$\frac{\partial f}{\partial t} = -\frac{a^2}{2} e^{ax - \frac{a^2}{2}t}$$

$$\frac{\partial f}{\partial x} = a\, e^{ax - \frac{a^2}{2}t}$$

$$\frac{\partial^2 f}{\partial x^2} = a^2 e^{ax - \frac{a^2}{2}t}$$

and we have

$$f(t, W_t) - f(0, W_0)$$

$$= \int_0^t \left(\frac{\partial f}{\partial s}(s, W_s)\, ds + \frac{\partial f}{\partial x}(s, W_s)\, dW_s \right.$$

$$\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, W_s)\, ds$$

which gives

$$e^{aW_t - \frac{a^2}{2}t} = \int_0^t \left(a e^{aW_s - \frac{a^2}{2}s} \right) dW_s .$$

because the terms in "ds" are exactly cancelling.