## Exam Martingales, Brownian motion and stochastic calculus (WI4430).

Tuesday 29th of January, 13:30-16:30.

Room: 3mE-IZ I/3mE-IZ M

- a) The exam has a theory part: questions 1 and 2, each on 10 points, and an exercise part (the remaining questions) on 20 points. The exercise part consists of 10 questions each on 2 points.
- c) No books, notes, calculators are allowed on the exam.
- d) The second reader of the exam is Dr. Ludolf Meester
- 1. State and prove the martingale convergence theorem. In case you prove the  $L^2$  version, then prove also the Kolmogorov-Doob inequality (8 points). Give an example of a martingale which does not converge almost surely (2 points).
- 2. a) Give the definition of Brownian motion. (2 points)
  - b) Show that Brownian motion is a Gaussian process (4 points).
  - c) Prove the formula for the quadratic variation of Brownian motion (4 points).
- 3. Let  $Y_i, i = 1, 2, ...$  be independent and identically distributed Bernouilli random variables with  $\mathbb{P}(Y_i = 1) = p \in (0, \frac{1}{2}), \mathbb{P}(Y_i = 0) = q = 1 p$ . Further define
  - i)  $\mathscr{F}_n = \sigma\{Y_i, i \leq n\}$  the natural filtration.
  - (ii)  $S_0 = 0, S_n = \sum_{i=1}^n (2Y_i 1).$
  - (iii)  $T = \inf\{n \ge 1 : Y_n = 1\}$
  - (iv) For  $a \in \mathbb{N}$ ,  $a \ge 1$ :  $\tau_a = \inf\{n \ge 1 : |S_n| \ge a\}$ 
    - a) Compute, for  $n \geq 3$  the conditional expectation

$$\mathbb{E}(S_n S_{n-1} | \mathscr{F}_{n-2})$$

b) Show that  $X_n := q^{-n}I(T > n)$  defines a martingale w.r.t.  $\mathscr{F}_n$ , where  $I(\cdot)$  denotes indicator function. (Hint: the event T > n is the event that all  $Y_i$  equal zero for  $i \in \{1, \ldots, n\}$ ).

- c) Show that the martingale  $X_n$  of item b) converges almost surely to zero and not in  $L^1$ .
- d) Show that

$$Z_n := \left(\frac{q}{p}\right)^{S_n}$$

is a  $\mathscr{F}_n$  martingale.

- e) Stop the martingale  $Z_n$  of item d) at the stopping time  $\tau_a$  to compute the probability  $\mathbb{P}(S_{\tau_a} = a)$ . You do not have to show that  $\tau_a$  is a finite stopping time, but you are asked to justify why you can exchange limits and expectations (if you do so).
- 4. Let  $\{W(t): t \geq 0\}$  be Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $\mathcal{F}_t = \sigma\{W(s): s \leq t\}$  denote its natural filtration. Further define, for a > 0, b < 0
  - i)  $\tau_a = \inf\{t > 0 : W(t) \ge a\}.$
  - ii)  $T_{a,b} = \inf\{t > 0 : W(t) \in \{a, b\}\}$
  - a) Compute, for  $0 \le s < t$  the conditional expectation

$$\mathbb{E}(W(t)^3 - 3tW(t)|\mathscr{F}_s)$$

Can you conclude that  $W(t)^3 - 3tW(t)$  is a  $\mathscr{F}_t$  martingale?

b) Stop an appropriate martingale to show that, for  $\lambda > 0$ 

$$\mathbb{E}(e^{-\lambda \tau_a}) = e^{-a\sqrt{2\lambda}}$$

You are allowed to use that  $\tau_a$  is a finite stopping time, but are asked to justify exchange of limits and expectations if applicable.

- c) Compute the probability  $\mathbb{P}(W(T_{a,b}) = b)$ . You are allowed to use that  $T_{a,b}$  is a finite stopping time, but are asked to justify exchange of limits and expectations if applicable.
- d) Integrated Brownian motion is defined as

$$X(t) = \int_0^t W(s)ds$$

Show that X(t) is normally distributed and compute its expectation and variance.

e) Show that, for p > 2 almost surely,

$$\lim_{N \to \infty} \sum_{i=1}^{N} (W(\frac{i}{N}) - W(\frac{i-1}{N}))^{p} = 0$$

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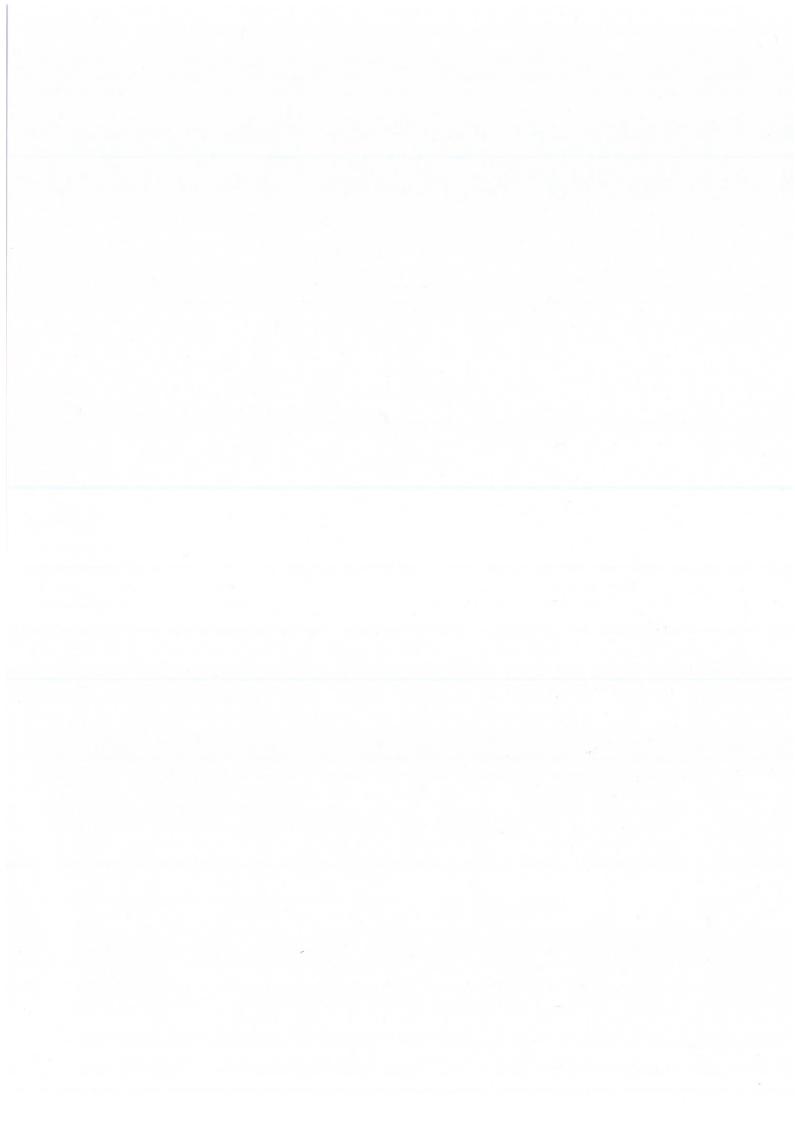
$$\mathbb{E}(S_n S_{n-1} | \mathscr{F}_{n-2})$$

b) Show that  $X_n := q^{-n}I(T > n)$  defines a martingale w.r.t.  $\mathscr{F}_n$ , where  $I(\cdot)$  denotes indicator function. (Hint: the event T > n is the event that all  $Y_i$  equal zero for  $i \in \{1, \ldots, n\}$ ).

$$\begin{array}{lll}
\vec{S}_{n-1} & S_{n-2} + \tilde{\gamma}_{n-1} + \tilde{\gamma}_{n} \\
S_{n-1} & S_{n-2} + \tilde{\gamma}_{n-1} \\
& \text{with } \tilde{\gamma}_{k} &= (2 \gamma_{k} - 1) \\
& \text{be have} \\
E(\tilde{\gamma}_{k}) &= (p-q) \\
E[\tilde{\gamma}_{k}^{2}] &= 1
\end{array}$$

$$\begin{split}
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E[\tilde{\gamma}_{k}^{2}] &= 1
\end{split}$$

$$\begin{split}
E(\left(S_{n-2} + \tilde{\gamma}_{n-1} + \tilde{\gamma}_{n}\right) \left(S_{n-2} + \tilde{\gamma}_{n-1}\right) \left(\tilde{J}_{n-2}\right) \\
&= S_{n-2} + S_{n-2} = \left[2\tilde{\gamma}_{n-1} + \tilde{\gamma}_{n}\right] \\
&+ E[\tilde{\gamma}_{n-1}^{2} + \tilde{\gamma}_{n}\tilde{\gamma}_{n-1}] + \tilde{\gamma}_{n} \\
&= S_{n-2} + S_{n-2} = \left[2\tilde{\gamma}_{n-1} + \tilde{\gamma}_{n}\right] \\
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&= S_{n-2}^{2} + S_{n-2}^{2} = \left[2\tilde{\gamma}_{$$



$$36. \quad I(T>n)$$

$$= \frac{n}{(1(1-\gamma_{1}))} (= I(\gamma_{1}=0,...,\gamma_{n}=0))$$

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Therefore if

$$X_n = g^{-n} \mathcal{I}(T > n)$$

we have

$$X_n = \prod_{i=1}^n Z_i \qquad Z_i = \left(\frac{1-Y_i}{g}\right)$$

This is clearly Fir measurable and

integrable (because  $\left|\frac{1-\gamma_i}{q}\right| \leq \frac{1}{q}$  so  $|X_n| \leq \frac{1}{q^n}$ )

To prove the martinale property:

$$= \left( \frac{1}{1} \frac{Z_i}{Z_i} \right) \left( E(Z_n) \right) = \chi_{n-1} . 1$$

rohere we used

$$\mathbb{E}\left(\frac{1-\chi_{n}}{q}\right) = \frac{1}{q} \cdot q = 1.$$



3c)  $X_n = \frac{1}{q^n} \int_{i=1}^{\infty} (i-\gamma_i)$ 

Notice that, on the set

have, for all WESZ'

Xn(w) -> 0. This set has probability

ove because in an independent signence

of zeros and ones with p= #(Yi)>0

there are elmost rurely infinitely many 1's

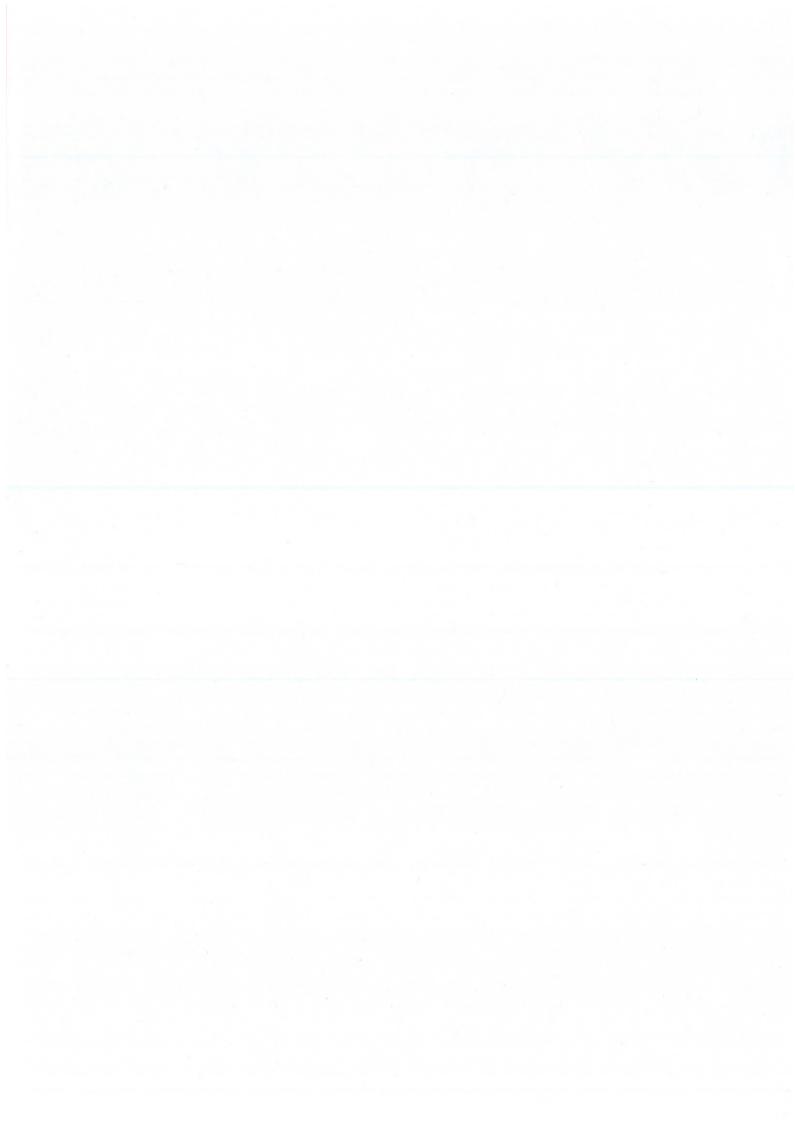
(by the strong law of large numbers).

Hence X4 ->0 a.s.

On the other hand, for all new,

 $E(X_n) = 1 \rightarrow 0$  as  $n \rightarrow \infty$ 

so the cannot converge to zero in L!



d) 
$$f_p^{Sn}$$
 is closely  $f_n$ -measurable  $f_p^{Sn} = (\frac{q}{p})^n r(\frac{r}{q})^n$  and integrable (because  $(\frac{q}{p})^{Sn} \leq (\frac{q}{p})^n r(\frac{r}{q})^n$ )

To show the montingale property notice that 
$$S_n = \sum_{i=1}^n (2\gamma_i - 1) := \sum_{i=1}^n \widetilde{\gamma}_i$$

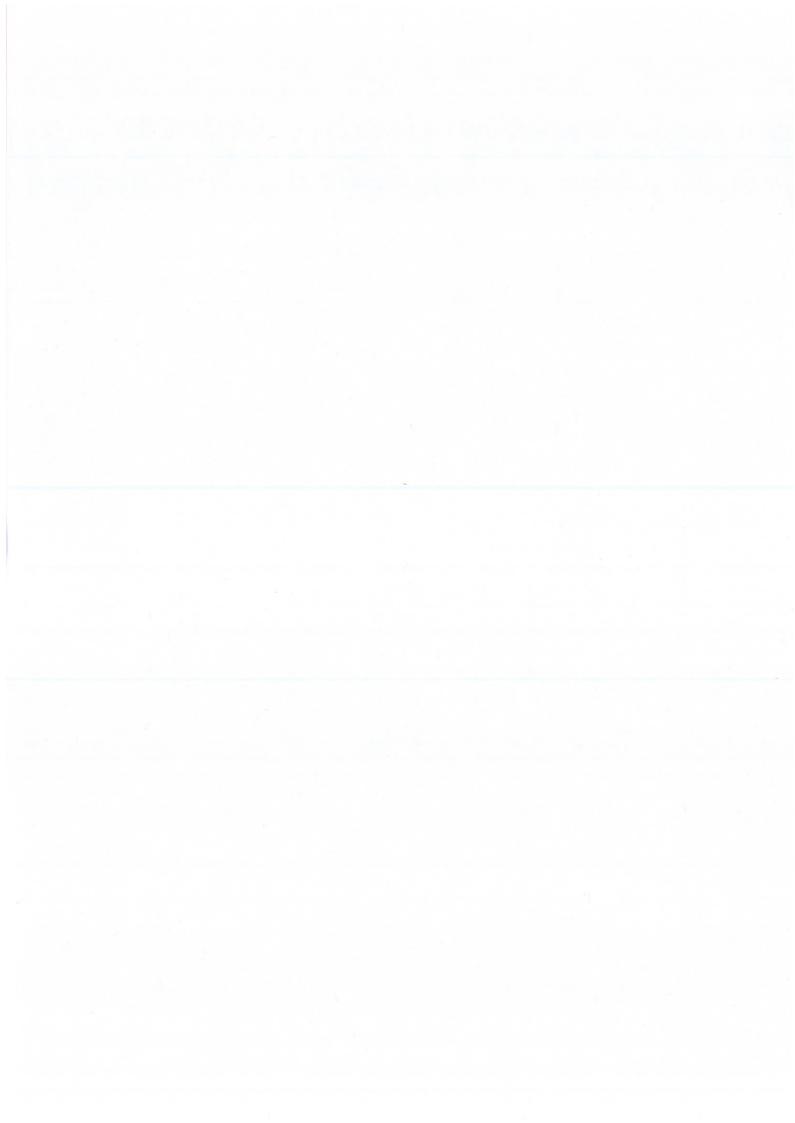
$$\mathbb{E}\left[\frac{q}{r}\right]^{S_n} \left| f_{n-1} \right]$$

Tok 
$$= (\frac{q}{p})^{S_{n-1}} E \left[ (\frac{q}{p})^{\frac{q}{y_n}} | f_{n-1} \right]$$

ind 
$$= \left(\frac{q}{r}\right)^{S_{n-1}} E\left[\left(\frac{q}{r}\right)^{\frac{\gamma}{\gamma_n}}\right]$$

$$= \left(\frac{q}{r}\right)^{S_{n-1}} \left[ \left(\frac{q}{p}\right) p + \frac{p}{q} q \right]$$

$$= \left(\frac{q}{p}\right)^{S_{n+1}}.$$



$$\overline{E}\left(\begin{pmatrix} 9 \\ p \end{pmatrix}^{S_{T_{\alpha}}} \wedge n \right) = 1 \quad \text{for all } n.$$

Now notice, since Is I sa we have

$$\left| \left( \frac{q}{p} \right)^{\frac{S}{4} \ln n} \right| \leq \left( \frac{q}{p} \right)^{\frac{q}{4}} \vee \left( \frac{q}{q} \right)^{\frac{q}{4}}$$

Therefore, we can bring in lines on his on the inside expectation, and using the finiteness of Ta, we get

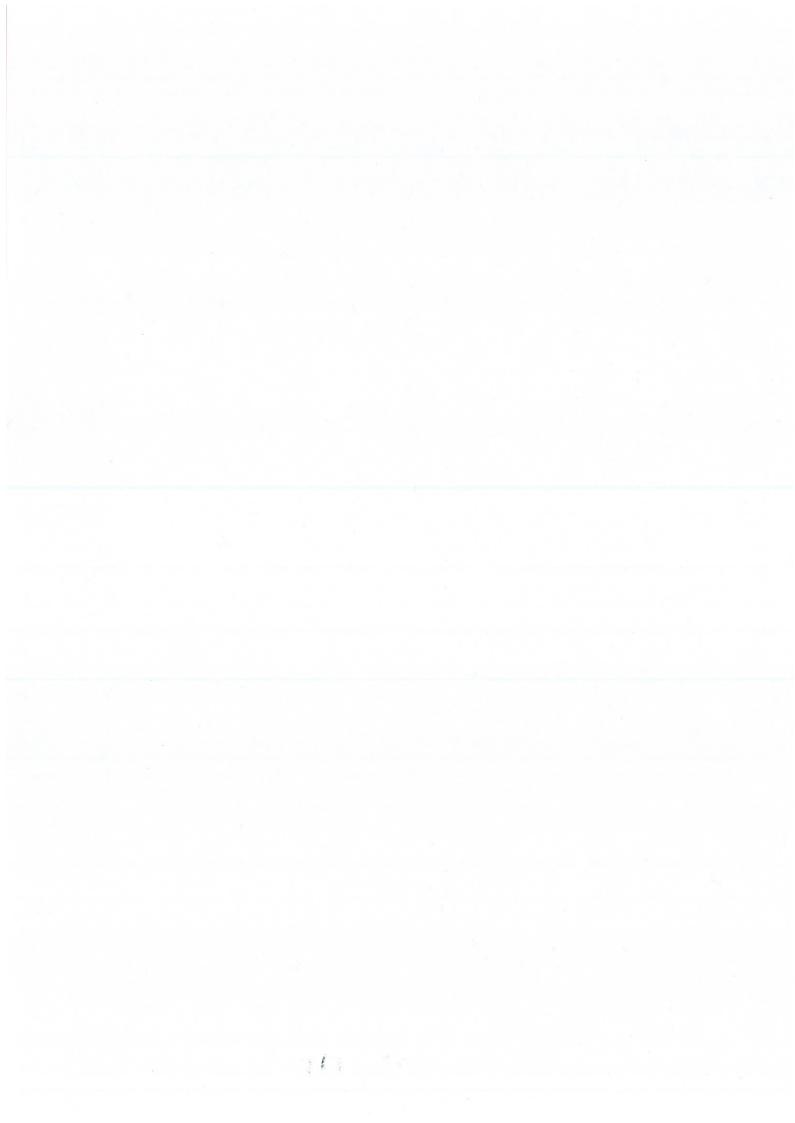
$$= \mathcal{I}\left(\left(\frac{q}{r}\right)^{S_{\tau_{\alpha}}}\right)$$



As a consequence

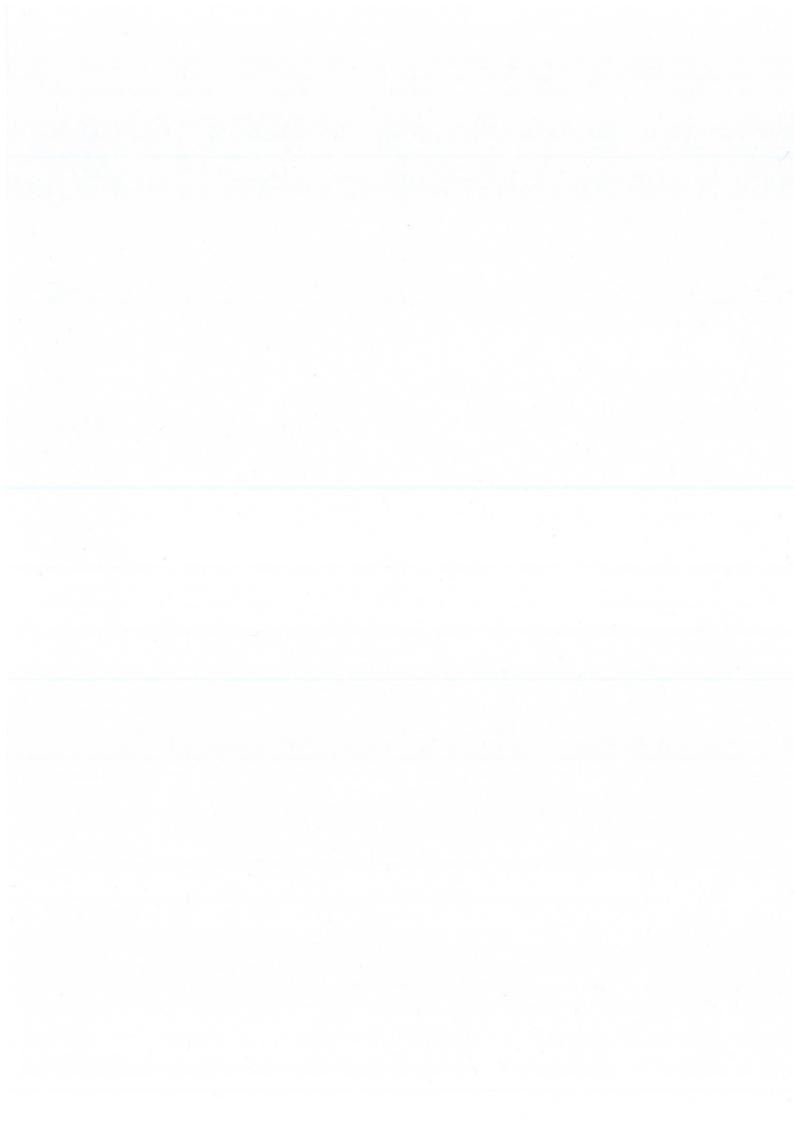
$$= \frac{(q)^{\alpha}}{(p)^{\alpha}} p(S_{t_{\alpha}} = \alpha) + \frac{(q)^{-\alpha}}{(p)^{-\alpha}} p(S_{t_{\alpha}} = -\alpha)$$

So
$$P\left(S_{t_{\alpha}} = a\right) = \frac{1 - \left(\frac{q}{p}\right)^{-q}}{\left(\frac{q}{p}\right)^{q} - \left(\frac{q}{p}\right)^{-q}}$$



4 a) clearly With - 3+ With is Ft - measurable and integrable (because https://o,t), all moments of W(t) exist). For SCt, compute E((W(B) + W(+)-W(B)) -3t(W(s) + W(t) - W(s)) / Fs / + 3 W/s) E( W/)-W/0/75) + \[ \big( \lambda (\forall 1) - \forall (\forall 1) \big) \big \frac{f\_s}{f\_s} \right) - 3t W(s) - 3t #((WH)-W(s))/\$) ind incr N(0, t-s) = W15) + 36/(s) (t-s) + 0 + 0 -3+ WIS) + 0

= W(S) - 35 W/S). So indeed, montigale.



b) Camider  $X_{t} = e^{\lambda W_{t} - \frac{\lambda^{2}}{2}t}$ 

We have, by optional sampling # + >0

 $\mathbb{E}\left(e^{\lambda W_{Ta} \Lambda t} - \frac{\lambda^2}{2} T_a \Lambda t\right) = 1$ 

Now use, for 2>0

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conveyence, continuity of WH), and

finitenen of stopping time, for all it >0

 $E(e^{\int_a^2 a - \frac{2}{2}} \tau_a) = 1$ 

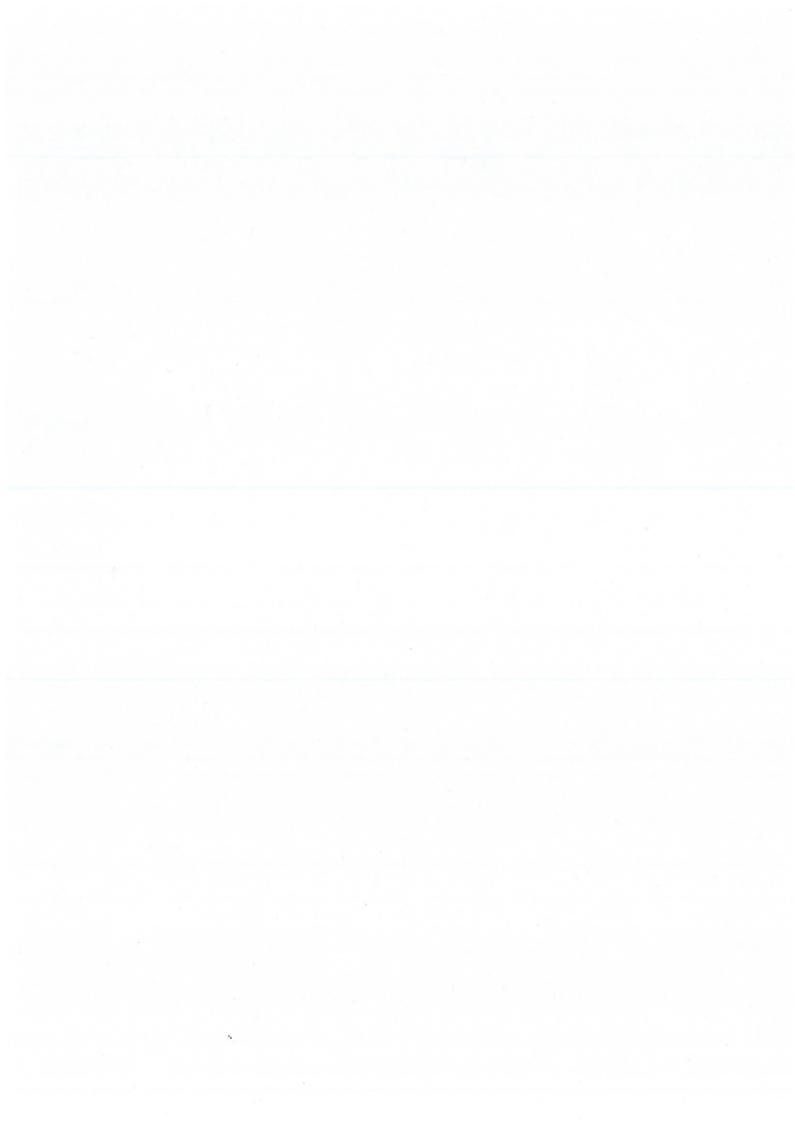
Putting  $\lambda = \sqrt{2\lambda}$ , we get  $e^{\sqrt{2}\lambda^2 a} = E(e^{-\lambda^2 ta}) = 1$ 



Wh. Carrider the montingal We have E(W( Ta, 6 1t)) = 0 Morcorer # | WITa, 6 Atr) | < 191 v 161 there, by dominated convergence lin E (W(Ta,61+1)) = E(W(Ta,6)) = 0 +-200 a P(W(Ta,6)=a) + & P(W(Ta,6)=6)=0

a (1-P(W(Ta,6)=B)) + 6 P(W(Ta,6)=B)=0

P(W\_Ta,6=6) = -a 6-a



d) Because WHI is continuous, the integral is the limit of its

Riemann sums, i.e., in L-sense:

Xd = lim I W/N/N
N-SN j=1

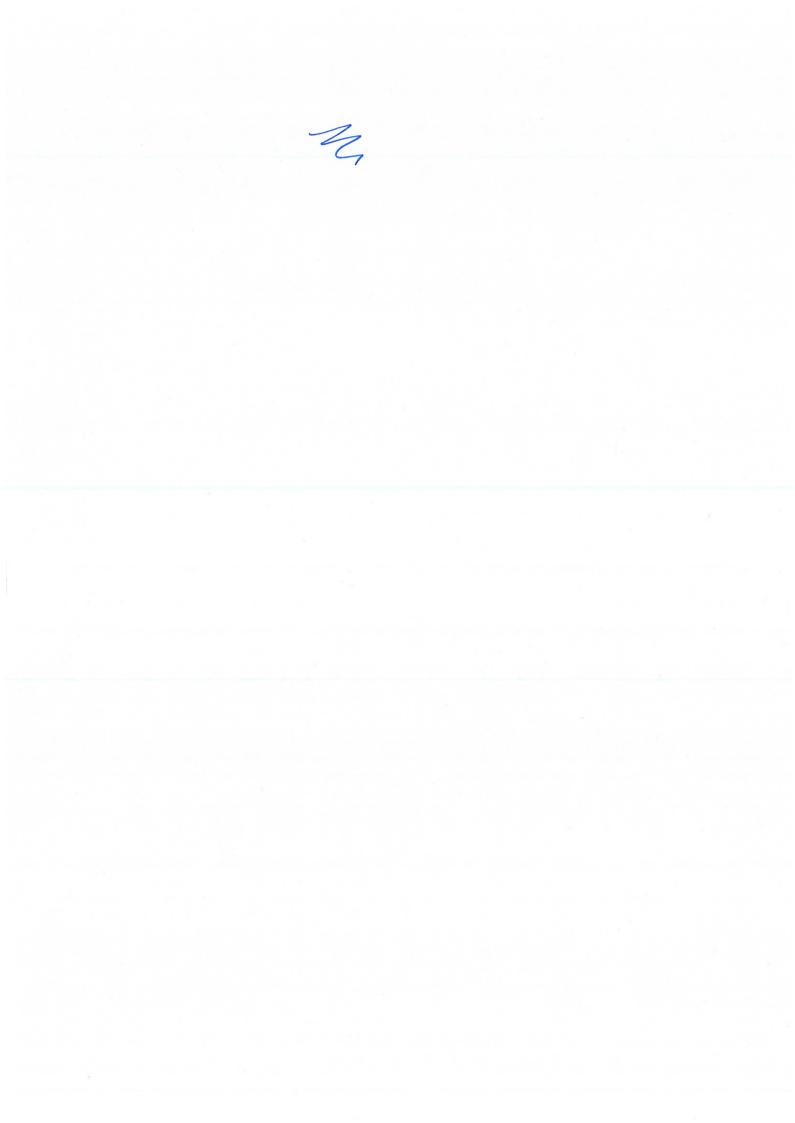
By the fact that WHI is a Gamman prien,  $\sum_{j=1}^{N}W(\frac{j+1}{N})\frac{t}{N}$  is normally distributed for all N, and a limit of normally distributed R.V. is normally

dirriented

E(X(t)) = 0  $Var(X(t)) = E(X(t)^2)$ 

= It (t E(W(s) W(n))) dsdn

=  $\int_0^t \int_0^t s ds ds = 2 \int_0^t \int_0^s ds = t^3/3$ .



Notice that

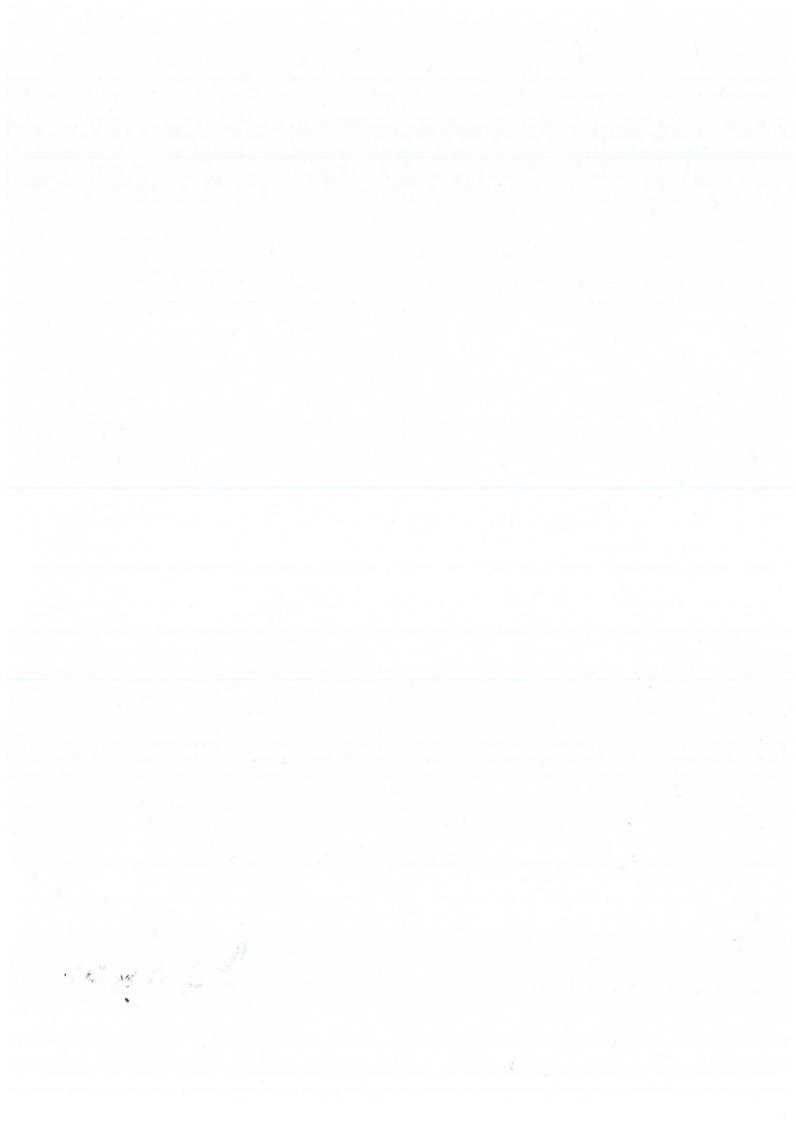
$$\sum_{i=1}^{N} (W(\hat{n}) - W(\hat{n}))^{p} = \sum_{i=1}^{N} N_{i}(0, n)$$

where No (0, 1/N) are independent normals with mean fevo and variance

1/5

$$P\left(\sum_{i=1}^{N}N_{i}\left(0,\frac{1}{N}\right)^{P}\geq\Sigma\right)$$

$$K = P \left( \sum_{i=1}^{N} N_i (o_{i})^{p} \geq \varepsilon N^{\frac{1}{2}} \right)$$



$$= P\left(\frac{1}{N}\sum_{i=1}^{N}N_{i}(0,1)^{2} \geq \varepsilon N^{2}\right)$$

$$\leq \frac{\mathbb{E}\left(\left(\frac{1}{N}\sum_{i=1}^{N}N_{i}\left(011\right)^{p}\right)^{K}\right)}{\mathbb{E}^{K}\left(N^{\frac{p}{2}-1}\right)^{K}}$$

Now notice that write law of large numbers + downexted convergence  $F\left[\left(\frac{1}{N}\sum_{i=1}^{N}N_{i}\left(o_{i1}\right)^{t}\right)^{K}\right]$ 

N-300 CPIK

Therefore, for X >0 large enough (NIZ-1) X is summable, and

 $\sum_{i=1}^{\infty} N_i(0, 1)^{\frac{1}{2}} \geq \varepsilon$ 

which imploy, by Borel- Cantelli

 $\sum_{i=1}^{N} \sqrt{i} \left( 0, \frac{1}{p} \right)^{p} \rightarrow 0 \quad \text{a.s.} \quad \text{as } N \rightarrow \infty.$ 

