Exam Martingales, Brownian motion and stochastic calculus (WI4430).
Tuesday 29th of January, 13:30-16:30.
Room: 3mE-IZ I/3mE-IZ M

a) The exam has a theory part: questions 1 and 2, each on 10 points, and an exercise part (the remaining questions) on 20 points. The exercise part consists of 10 questions each on 2 points.

c) No books, notes, calculators are allowed on the exam.

d) The second reader of the exam is Dr. Ludolf Meester

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1. State and prove the martingale convergence theorem. In case you prove the L^2 version, then prove also the Kolmogorov-Doob inequality (8 points). Give an example of a martingale which does not converge almost surely (2 points).

2. a) Give the definition of Brownian motion. (2 points)
   b) Show that Brownian motion is a Gaussian process (4 points).
   c) Prove the formula for the quadratic variation of Brownian motion (4 points).

3. Let $Y_i, i = 1, 2, \dots$ be independent and identically distributed Bernouilli random variables with $\mathbb{P}(Y_i = 1) = p \in (0, \frac{1}{2}), \mathbb{P}(Y_i = 0) = q = 1 - p$. Further define

   i) $\mathscr{F}_n = \sigma\{Y_i, i \leq n\}$ the natural filtration.
   (ii) $S_0 = 0, S_n = \sum_{i=1}^n (2Y_i - 1)$.
   (iii) $T = \inf\{n \geq 1 : Y_n = 1\}$
   (iv) For $a \in \mathbb{N}, a \geq 1$: $\tau_a = \inf\{n \geq 1 : |S_n| \geq a\}$

   a) Compute, for $n \geq 3$ the conditional expectation
   $$\mathbb{E}(S_n S_{n-1} | \mathscr{F}_{n-2})$$
   b) Show that $X_n := q^{-n} I(T > n)$ defines a martingale w.r.t. $\mathscr{F}_n$, where $I(\cdot)$ denotes indicator function. (Hint: the event $T > n$ is the event that all Y_i equal zero for $i \in \{1, \dots, n\}$).

c) Show that the martingale X_n of item b) converges almost surely to zero and not in L^1.

d) Show that

$$Z_n := \left(\frac{q}{p}\right)^{S_n}$$

is a $\mathscr{F}_n$ martingale.

e) Stop the martingale Z_n of item d) at the stopping time τ_a to compute the probability $\mathbb{P}(S_{\tau_a} = a)$. You do not have to show that τ_a is a finite stopping time, but you are asked to justify why you can exchange limits and expectations (if you do so).

4. Let $\{W(t) : t \geq 0\}$ be Brownian motion defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{F}_t = \sigma\{W(s) : s \leq t\}$ denote its natural filtration. Further define, for $a > 0, b < 0$

i) $\tau_a = \inf\{t > 0 : W(t) \geq a\}$.

ii) $T_{a,b} = \inf\{t > 0 : W(t) \in \{a, b\}\}$

a) Compute, for $0 \leq s < t$ the conditional expectation

$$\mathbb{E}(W(t)^3 - 3tW(t)|\mathscr{F}_s)$$

Can you conclude that $W(t)^3 - 3tW(t)$ is a $\mathscr{F}_t$ martingale?

b) Stop an appropriate martingale to show that, for $\lambda > 0$

$$\mathbb{E}(e^{-\lambda\tau_a}) = e^{-a\sqrt{2\lambda}}$$

You are allowed to use that τ_a is a finite stopping time, but are asked to justify exchange of limits and expectations if applicable.

c) Compute the probability $\mathbb{P}(W(T_{a,b}) = b)$. You are allowed to use that $T_{a,b}$ is a finite stopping time, but are asked to justify exchange of limits and expectations if applicable.

d) Integrated Brownian motion is defined as

$$X(t) = \int_0^t W(s)ds$$

Show that $X(t)$ is normally distributed and compute its expectation and variance.

e) Show that, for $p > 2$ almost surely,

$$\lim_{N\to\infty} \sum_{i=1}^{N} (W(\tfrac{i}{N}) - W(\tfrac{i-1}{N}))^p = 0$$

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Solutions Exercises

3 a)

$$S_n = (S_{n-2} + \tilde{Y}_{n-1} + \tilde{Y}_n)$$

$$S_{n-1} = (S_{n-2} + \tilde{Y}_{n-1})$$

with $\tilde{Y}_k = (2Y_k - 1)$

we have

$$\mathbb{E}(\tilde{Y}_k) = (p-q)$$

$$\mathbb{E}[\tilde{Y}_k^2] = 1$$

$$\mathbb{E}[(S_{n-2} + \tilde{Y}_{n-1} + \tilde{Y}_n)(S_{n-2} + \tilde{Y}_{n-1}) / \mathcal{F}_{n-2}]$$

$$\overset{TOK}{=} S_{n-2}^2 + S_{n-2}\,\mathbb{E}[2\tilde{Y}_{n-1} + \tilde{Y}_n / \mathcal{F}_{n-2}]$$

$$+ \mathbb{E}[\tilde{Y}_{n-1}^2 + \tilde{Y}_n \tilde{Y}_{n-1} / \mathcal{F}_{n-2}]$$

$$\overset{IND}{=} S_{n-2}^2 + S_{n-2}\,\mathbb{E}(2\tilde{Y}_{n-1} + \tilde{Y}_n)$$

$$+ \mathbb{E}(\tilde{Y}_{n-1}^2 + \tilde{Y}_n \tilde{Y}_{n-1})$$

$$= S_{n-2}^2 + S_{n-2}\,3(p-q) + 1 + (p-q)^2.$$

3b. $I(T > n)$

$$= \prod_{i=1}^{n} (1 - Y_i) \quad (= I(Y_1 = 0, \dots, Y_n = 0))$$

Therefore if

$$X_n = 9^{-n} \, I(T > n)$$

we have

$$X_n = \prod_{i=1}^{n} Z_i \qquad Z_i = \left(\frac{1 - Y_i}{9} \right)$$

This is clearly $\mathcal{F}_n$- measurable and integrable (because $\left| \frac{1 - Y_i}{9} \right| \leq \frac{1}{9}$ so $|X_n| \leq \frac{1}{9^n}$)

To prove the martingale property:

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \overset{\text{Tok}}{=} \prod_{i=1}^{n-1} Z_i \; \mathbb{E}(Z_n \mid \mathcal{F}_{n-1})$$

$$\overset{\text{IND}}{=} \left(\prod_{i=1}^{n-1} Z_i \right) \left(\mathbb{E}(Z_n) \right) = X_{n-1} \cdot 1$$

where we used

$$\mathbb{E}(Z_n) = \mathbb{E}\left(\frac{1 - Y_n}{9} \right) = \frac{1}{9} \cdot 9 = 1.$$

3c) $X_n = \frac{1}{q^n} \prod_{i=1}^{n} (1 - Y_i)$

Notice that, on the set

$\Omega' = \{\omega \in \Omega : \exists n : Y_n(\omega) = 1\}$ we

have, for all $\omega \in \Omega'$

$X_n(\omega) \longrightarrow 0$. This set has probability

one because in an independent sequence

of zeros and ones with $p = \mathbb{E}(Y_i) > 0$

there are almost surely infinitely many 1's

(by the strong law of large numbers).

Hence $X_n \longrightarrow 0$ a.s.

On the other hand, for all $n \in \mathbb{N}$,

$\mathbb{E}(X_n) = 1 \not\rightarrow 0$ as $n \to \infty$

so X_n cannot converge to zero in L^1!

d) $\left(\frac{q}{p}\right)^{S_n}$ is clearly $\mathcal{F}_n$- measurable and integrable (because $\left(\frac{q}{p}\right)^{S_n} \le \left(\frac{q}{p}\right)^n \vee \left(\frac{p}{q}\right)^n$)

To show the martingale property notice that $S_n = \sum_{i=1}^{n} (2Y_i - 1) := \sum_{i=1}^{n} \tilde{Y}_i$

so

$$\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n} \middle| \mathcal{F}_{n-1}\right]$$

$$\overset{\text{TOK}}{=} \left(\frac{q}{p}\right)^{S_{n-1}} \mathbb{E}\left[\left(\frac{q}{p}\right)^{\tilde{Y}_n} \middle| \mathcal{F}_{n-1}\right]$$

$$\overset{\text{IND}}{=} \left(\frac{q}{p}\right)^{S_{n-1}} \mathbb{E}\left[\left(\frac{q}{p}\right)^{\tilde{Y}_n}\right]$$

$$= \left(\frac{q}{p}\right)^{S_{n-1}} \left[\left(\frac{q}{p}\right) p + \frac{p}{q} q\right]$$

$$= \left(\frac{q}{p}\right)^{S_{n-1}}.$$

e) By optional sampling we have

$$\mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{\tau_a \wedge n}} \right) = 1 \quad \text{for all } n.$$

Now notice, since $|S_{\tau_a \wedge n}| \leq a$ we have

$$\left| \left(\frac{q}{p}\right)^{S_{\tau_a \wedge n}} \right| \leq \left(\frac{q}{p}\right)^a \vee \left(\frac{p}{q}\right)^a$$

Therefore, we can bring in $\lim_{n \to \infty}$ inside expectation, and using the finiteness of τ_a, we get

$$1 = \lim_{n \to \infty} \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{\tau_a \wedge n}} \right)$$

$$= \mathbb{E}\left(\lim_n \left(\frac{q}{p}\right)^{S_{\tau_a \wedge n}} \right)$$

$$= \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{\tau_a}} \right)$$

As a consequence

$$1 = \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{\tau_a}} \right)$$

$$= \left(\frac{q}{p}\right)^{a} P(S_{\tau_a} = a) + \left(\frac{q}{p}\right)^{-a} P(S_{\tau_a} = -a)$$

$$= \left(\frac{q}{p}\right)^{a} P(S_{\tau_a} = a) + \left(\frac{q}{p}\right)^{-a} \left(1 - P(S_{\tau_a} = a)\right)$$

$$= 1$$

So

$$P(S_{\tau_a} = a) = \frac{1 - \left(\frac{q}{p}\right)^{-a}}{\left(\frac{q}{p}\right)^{a} - \left(\frac{q}{p}\right)^{-a}}$$

4a) clearly $W(t)^3 - 3tW(t)$ is $\mathcal{F}_t$-measurable and integrable (because $W(t) \sim N(0,t)$, all moments of $W(t)$ exist).

For $s<t$, compute

$$\mathbb{E}\Big(\big(W(s) + W(t) - W(s)\big)^3 - 3t\big(W(s) + W(t) - W(s)\big) \,\Big|\, \mathcal{F}_s \Big)$$

TOK

$$= W(s)^3 + 3W(s)\,\mathbb{E}\big((W(t)-W(s))^2 \mid \mathcal{F}_s\big)$$
$$+ 3W(s)^2\,\mathbb{E}\big(W(t)-W(s) \mid \mathcal{F}_s\big)$$
$$+ \mathbb{E}\big((W(t)-W(s))^3 \mid \mathcal{F}_s\big)$$
$$- 3tW(s) - 3t\,\mathbb{E}\big((W(t)-W(s)) \mid \mathcal{F}_s\big)$$

indep incr $N(0,t-s)$

$$= W(s)^3 + 3W(s)(t-s) + 0 + 0$$
$$- 3tW(s) + 0$$
$$= W(s)^3 - 3sW(s).$$

So indeed, martingale.

b) Consider

$$X_t = e^{\lambda W_t - \frac{\lambda^2}{2} t}$$

We have, by optional sampling $\forall t > 0$

$$\mathbb{E}\left(e^{\lambda W_{\tau_a \wedge t} - \frac{\lambda^2}{2} \tau_a \wedge t} \right) = 1$$

Now use, for $\lambda > 0$

$$e^{\lambda W_{\tau_a \wedge t} - \lambda^2/2 \, \tau_a \wedge t}$$

$\leq e^{\lambda a}$, so using dominated convergence, continuity of $W(t)$, and finiteness of stopping time, for all $\tilde{\lambda} > 0$

$$\mathbb{E}\left(e^{\tilde{\lambda} a - \tilde{\lambda}^2/2 \, \tau_a} \right) = 1$$

Putting $\tilde{\lambda} = \sqrt{2\lambda}$, we get

$$e^{\sqrt{2\lambda}\, a} \, \mathbb{E}\left(e^{-\lambda \tau_a} \right) = 1$$

c) Consider the martingale $W(t)$.

We have

$$\mathbb{E}(W(T_{a,b} \wedge t)) = 0$$

Moreover

~~$\mathbb{E}$~~ $|W(T_{a,b} \wedge t)| \leq |a| \vee |b|$

Hence, by dominated convergence

$$\lim_{t \to \infty} \mathbb{E}(W(T_{a,b} \wedge t)) = \mathbb{E}(W(T_{a,b})) = 0$$

$$a\, P(W(T_{a,b}) = a) + b\, P(W(T_{a,b}) = b) = 0$$

$$a\,(1 - P(W(T_{a,b}) = b)) + b\, P(W(T_{a,b}) = b) = 0$$

So

$$P(W_{T_{a,b}} = b) = \frac{-a}{b-a} = \frac{a}{a-b}$$

d) Because $W(t)$ is continuous, the integral is the limit of its Riemann sums, i.e., in L^2-sense:

$$X(t) = \lim_{N\to\infty} \sum_{j=1}^{N} W\left(\frac{jt}{N}\right)\frac{t}{N}$$

By the fact that $W(t)$ is a Gaussian process, $\sum_{j=1}^{N} W\left(\frac{jt}{N}\right)\frac{t}{N}$ is normally distributed for all N, and a limit of normally distributed R.V. is normally distributed.

$$\mathbb{E}(X(t)) = 0$$

$$\mathrm{Var}(X(t)) = \mathbb{E}(X(t)^2)$$

$$= \int_0^t \int_0^t \mathbb{E}(W(s)\,W(r))\,ds\,dr$$

$$= \int_0^t \int_0^t s \wedge r \; ds\,dr = 2 \int_0^t \int_0^s r\,dr = t^3/3.$$

Notice that

$$\sum_{i=1}^{N} \left(W\left(\tfrac{i}{N}\right) - W\left(\tfrac{i-1}{N}\right) \right)^p = \sum_{i=1}^{N} N_i\left(0, \tfrac{1}{N}\right)^p$$

where $N_i(0, 1/N)$ are independent normals with mean zero and variance $1/N$.

Notice $N_i\left(0, \frac{1}{N}\right) = \frac{1}{\sqrt{N}} N_i(0,1)$

so for $\varepsilon > 0$

$$P\left(\sum_{i=1}^{N} N_i\left(0, \tfrac{1}{N}\right)^p \geq \varepsilon \right)$$

$$= P\left(\sum_{i=1}^{N} N_i(0,1)^p \geq \varepsilon N^{\frac{p}{2}} \right)$$

$$= P\left(\frac{1}{N} \sum_{i=1}^{N} N_i(0,1)^p \geq \varepsilon N^{\frac{p}{2}-1} \right)$$

$$\leq \frac{\mathbb{E}\left(\left(\frac{1}{N} \sum_{i=1}^{N} N_i(0,1)^p \right)^K \right)}{\varepsilon^K \left(N^{\frac{p}{2}-1} \right)^K}$$

Now notice that using law of large numbers + dominated convergence

$$\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^{N} N_i(0,1)^p \right)^K \right] \xrightarrow{N \to \infty} C_{p,K}$$

Therefore, for $K > 0$ large enough $\left(\frac{1}{N^{p/2-1}} \right)^K$ is summable, and

$$\sum_{N=1}^{\infty} P\left(\sum_{i=1}^{N} N_i(0,\tfrac{1}{N})^p \geq \varepsilon \right) < \infty$$

which implies, by Borel-Cantelli

$$\sum_{i=1}^{N} N_i(0,\tfrac{1}{N})^p \to 0 \quad \text{a.s.} \quad \text{as } N \to \infty.$$