Exam Stochastic Processes WI 4202.

30 january 2015, 14:00-17:00 EWI lecture hall F/G.

No books or notes allowed.

Responsible of the course: Prof. F. Redig

Second reader exam: Dr. Ludolf Meester

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a) The exam consists of two theory questions, each on 10 points, followed by exercises. The exercises consist of 10 small questions each on 2 points.

b) The end score is computed as explained on the blackboard page. Course grade is the final exam grade f or $0.6f + 0.4h$ (with h average homework grade), whichever is larger, provided $f \geq 5$.

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1 Theory Questions.

1) State and prove the martingale convergence theorem. If you give the L^2 proof (the proof from the book), then prove also Kolmogorov's maximal inequality. If you give the proof based on upcrossings, then prove also the Doob's upcrossing inequality.

2) a) Give the definition of Brownian motion.
   b) Derive the explicit formula for the probability density of the first hitting time of $a > 0$ for Brownian motion.

2 Exercises.

1) $\{X_i, i \in \mathbb{N}\}$ are independent and identically distributed random variables with a standard normal distribution (i.e., normally distributed with mean zero and variance 1). Furthermore, let $\{a_n, n \in \mathbb{N}\}$ be sequence of real numbers. In the whole exercise you are allowed to use the expression for the moment generating function of a normal random variable Y with mean μ and variance σ^2:

$$\mathbb{E}(e^{tY}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

a) Compute the conditional expectation

$$\mathbb{E}\left(X_1 + X_2 + e^{X_1+X_2+X_3} \mid X_1, X_2\right)$$

b) Show that $\{M_n, n \geq 1\}$ defined via

$$M_n = \sum_{i=1}^{n} a_i X_i$$

is a martingale.

c) Show that if $\sum_{i=1}^{\infty} a_i^2 < \infty$ then the martingale of item b) satisfies the conditions of the martingale convergence theorem. Conclude that in that case the series

$$\sum_{i=1}^{\infty} a_i X_i$$

converges with probability 1.

d) Show that $\{Z_n, n \geq 1\}$ defined via

$$Z_n = e^{\sum_{i=1}^{n} a_i X_i - \frac{1}{2}\sum_{i=1}^{n} a_i^2}$$

is a martingale. Does the martingale convergence theorem apply to this martingale?

e) Let Z_n be as in item d). Define n new random variables $Y_1, \ldots, Y_n$ via

$$\mathbb{E}(f(Y_1, \ldots, Y_n)) = \mathbb{E}(Z_n f(X_1, \ldots, X_n))$$

for all f such that the expectations in the right hand side exist. Show that $Y_1, \ldots, Y_n$ thus defined are independent and normally distributed with mean $\mathbb{E}(Y_i) = a_i$ and variance $Var(Y_i) = 1$. Hint: it is sufficient to show that $(Y_1, \ldots, Y_n)$ has the correct multivariate moment generating function, i.e., that

$$\mathbb{E}(e^{\sum_{i=1}^{n} t_i Y_i}) = e^{\sum_{i=1}^{n}\left(t_i a_i + \frac{t_i^2}{2}\right)}.$$

for all $t_1, \ldots, t_n \in \mathbb{R}$.

2) Let $\{W_t, t \geq 0\}$ denote Brownian motion, and let $\{N_t, t \geq 0\}$ denote rate one Poisson process (i.e., N_t is Poisson with parameter t) which is furthermore independent from $\{W_t, t \geq 0\}$. You are allowed to use that for a Poisson random variable N with parameter λ one has $\mathbb{E}(N) = \lambda$, $Var(N) = \lambda$, $\mathbb{E}(e^{sN}) = e^{\lambda(e^s-1)}$, $s \in \mathbb{R}$.

a) Show that

$$\mathbb{E}(W_{N_t}^2) = t$$

b) Show that $\{Z_t, t \geq 0\}$ defined by

$$Z_t = W_t^2 - t$$

is a martingale.

c) Define the exit time of the interval $[-a, a]$ by

$$\tau = \inf\{t \geq 0 : |W_t| > a\}$$

Show that τ is a finite stopping time.

d) Let τ be as in item c). In this item you are allowed to use

1) that $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$ are both finite.
2) Your are also allowed the martingale of item b) and
3) also you are allowed to use that $M_t = W_t^4 - 6tW_t^2 + 3t^2$ is a martingale (i.e., you do not have to prove this).

Show then that

$$\mathbb{E}(\tau) = a^2, \mathbb{E}(\tau^2) = \frac{5a^4}{3}$$

If you use the martingale stopping theorem, you should argue why you are allowed to use it.

e) In this item you are allowed to use that $N_t - t$, and $(N_t - t)^2 - t$ are martingales. Define

$$\tau_n = \inf\{t \geq 0 : N_t \geq n\}$$

In this item, you do not have to verify the conditions of the martingale stopping theorem, i.e., you can assume that they are satisfied. Prove the following equalities using martingale stopping:

$$\mathbb{E}((\tau_n - n)^2) = \mathbb{E}(\tau_n) = n$$