Final Mathematical Structures AM1010
Monday January 27, 2020, 9:00-12:00

No calculators allowed. Write the solutions in the fields provided. The grade is (score+8)/8.

Exercise [ ] continued (extra space)

1. Consider the statement $(p \vee q) \Rightarrow (p \wedge q)$.

   (a) Give the truth table of this statement. 3

   (b) Is this statement a tautology? Explain your answer! 1

   Yes/No because

2. What is the error in the following proof? Give the line number and explain what goes wrong. 4

   Consider the relation on $\mathbb{R}$ defined by xRy if and only if $xy \geq 0$. We will show the relation is transitive

   i) Suppose xRy and yRz hold,
   ii) Then $xy \geq 0$ and $yz \geq 0$.
   iii) Therefore $xy \cdot yz \geq 0$.
   iv) Note that $y^2 \geq 0$ for all $y \in \mathbb{R}$,
   v) Thus we conclude $xz \geq 0$.
   vi) Hence xRz holds as well.

3. Formulate the completeness axiom for the real numbers. 2

(Give the one from this course, not the one from AM2090: Real analysis.)

4. (a) Complete the definition of a Cauchy sequence. 2

A sequence (s_n) is Cauchy if

(b) Prove that Cauchy sequences of real numbers converge. You may use the fact that a Cauchy sequence is bounded, and the theorem of Bolzano-Weierstrass. 6

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a decreasing function (so if $x < y$, then $f(x) > f(y)$) and A is a bounded set.

(a) Give an explicit example of a decreasing function $f : \mathbb{R} \to \mathbb{R}$ and a bounded set A for which the strict inequality $\sup(f(A)) < f(\inf(A))$ holds. Also provide the values of $\sup(f(A))$ and $\inf(A)$, but you don't need to show your calculations for these. 5

$f(x) =$

$A =$

$\inf(A) =$

$f(\inf(A)) =$

$\sup(f(A)) =$

(b) Show that in general $\sup(f(A)) \leq f(\inf(A))$. 6

The axioms of an ordered field as applied to $\mathbb{R}$ are

A1 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x + y = w + z$;
A2 $\forall x, y \in \mathbb{R} : x + y = y + x$;
A3 $\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$;
A4 $\exists 0 : \forall x \in \mathbb{R} : x + 0 = x$ and this 0 is unique;
A5 $\forall x \in \mathbb{R} : \exists (-x) \in \mathbb{R} : x + (-x) = 0$ and $(-x)$ is unique;
M1 $\forall x, y \in \mathbb{R} : x \cdot y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x \cdot y = w \cdot z$;
M2 $\forall x, y \in \mathbb{R} : x \cdot y = y \cdot x$;
M3 $\forall x, y, z \in \mathbb{R} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
M4 $\exists 1 \neq 0 : \forall x \in \mathbb{R} : x \cdot 1 = x$ and this 1 is unique;
M5 $\forall x \neq 0 : \exists (1/x) \in \mathbb{R} : x \cdot (1/x) = 1$ and $(1/x)$ is unique;
DL $\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = x \cdot y + x \cdot z$;
O1 For all $x, y \in \mathbb{R}$ exactly one of $x = y$, $x > y$, holds $x < y$;
O2 $\forall x, y, z \in \mathbb{R} : x < y \wedge y < z \Rightarrow x < z$;
O3 $\forall x, y, z \in \mathbb{R} : x < y \Rightarrow x + z < y + z$;
O4 $\forall x, y, z \in \mathbb{R} : x < y \wedge 0 < z \Rightarrow xz < yz$.

6. Let $x, y \in \mathbb{R}$. Show that if $x < y$ and $0 < x + y$ then $x \cdot x < y \cdot y$ using only the axioms. At every step, specify which axioms you use[1]. 6

[1] $0 < x + y$ of course corresponds to $-y < x$, but showing that takes quite a few steps usings axioms, so you don't have to do that.

7. Consider the (convergent) series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)}$.

(a) Give an explicit formula for the partial sum $s_n = \sum_{k=0}^{n} \frac{1}{(2k+1)(2k+5)}$ and prove it using induction. 7

$s_n =$

(b) Determine the value of the series, and show that your result is correct. 2

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)} =$

8. Determine for the following series whether they are absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n+1}{n}}$ 3

(b) $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5n+2}$. 4

9. Determine the radius of convergence and the endpoints of the interval of convergence for the series 5

$$\sum_{n=1}^{\infty} \frac{9^n(2x+1)^{2n}}{n}.$$

You don't have to determine whether or not the series converges at the endpoints of the interval of convergence.

Write the results in the boxes after you calculated them; use the space underneath to explain your results.

$R =$ [ ]

Interval of convergence starts at [ ] and ends at [ ]

10. Give two bounded sequences (s_n) and (t_n) such that 4

$$\limsup(s_n - t_n) \neq \limsup s_n - \liminf t_n.$$

Show your example is correct by calculating the relevant quantities. You don't have to prove that the value of $\limsup s_n$ is what you say it is, etc.

$s_n =$

$t_n =$

$\limsup s_n =$

$\liminf t_n =$

$\limsup(s_n - t_n) =$

11. Suppose $\sum a_n$ and $\sum b_n$ are absolutely convergent. Show that $\sum a_n b_n$ is also absolutely convergent. 6

12. Suppose (a_n) is a decreasing sequence of positive terms and $\sum a_n$ is convergent, then $\lim na_n = 0$. 6

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