Test 1 Mathematical Structures AM1010
Wednesday October 2, 2019, 9:00-10:00

TU Delft

No calculators allowed. Write the solutions in the fields provided. The grade is (score+4)/4.

1a Write down the truth table for the expression $(q \Rightarrow p) \wedge (\sim p)$. 3

Solution.

p	q	$(q \Rightarrow p)$	$\wedge$	$(\sim p)$
T	T	T	F	F
T	F	T	F	F
F	T	F	F	T
F	F	T	T	T

☐

1b Give a statement R in terms of p and q, expressed without using $\Rightarrow$ and $\wedge$ such that $(q \Rightarrow p) \wedge (\sim p) \Leftrightarrow R$ is a tautology. For example R could be $(\sim q) \vee p$. 2

You don't have to explain your answer.

Solution. $\sim (p \vee q)$

The statement is true only when both p and q are false, so it is equivalent to $(\sim p) \wedge (\sim q)$. This is not an allowed answer due to the $\wedge$, but it is equivalent to $\sim (p \vee q)$. ☐

2a Give the definition of a partition of a set. 2

A partition of the set S is ...

Solution. See Definition 2.2.12

a collection $\mathcal{P}$ of subsets of S such that

- They are all non-empty ($\emptyset \notin \mathcal{P}$)
- They are pairwise disjoint. ($A, B \in \mathcal{P}$ with $A \neq B$, then $A \cap B = \emptyset$)
- They cover S (For each $x \in S$ there is $A \in \mathcal{P}$ with $x \in A$). ☐

2b Give an example of a partition of the set $S = \{5, 8, 12\}$. 1

Solution. There are only 5 partitions of this set (you only needed to give one):

$$\{\{5,8,12\}\}, \quad \{\{5,8\},\{12\}\}, \quad \{\{5,12\},\{8\}\}, \quad \{\{8,12\},\{5\}\}, \quad \{\{5\},\{8\},\{12\}\}$$

☐

3 Let A, B, and C be sets. Show that $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$. 6

Solution. Let $x \in (A \setminus C) \cup (B \setminus C)$. Then $x \in A \setminus C$ or $x \in B \setminus C$. Thus either $x \in A$ and $x \notin C$ or $x \in B$ and $x \not\mathit{inC}$. Thus in both cases $x \notin C$. Moreover in both cases $x \in A$ or $x \in B$, thus $x \in A \cup B$. We conclude that $x \in (A \cup B) \setminus C$. Thus the inclusion $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$ holds.

Remark: In this case we even have equality. □

4 We define a relation R on $\mathbb{N}$ as nRm iff there are odd integers p and q such that $\frac{n}{m} = \frac{p}{q}$. You may assume that this relation is transitive. Remember to give a proof for all your answers.

4a Is the relation R reflexive? 2

Solution. Yes.

Let $n \in \mathbb{N}$ be arbitrary. Then $\frac{n}{n} = \frac{1}{1}$ is a quotient of the desired form, so $1R1$ holds. □

4b Is the relation R symmetric? 3

Solution. Yes.

Let $n, m \in \mathbb{N}$ be arbitrary and suppose nRm holds. Then $\frac{n}{m} = \frac{p}{q}$ for some odd integers p and q, and thus $\frac{m}{n} = \frac{q}{p}$ can also be written as a quotient of two odd numbers. Thus mRn holds as well. □

4c Is the relation R an equivalence relation? If so, give a simple expression for the equivalence class E_1. 2

Solution. Yes, it is both reflexive, symmetric and transitive.

The equivalence class E_1 is the set of elements equivalent to 1. Thus $xR1$ must hold and $x = \frac{p}{q}$ must be the quotient of two odd integers. This is true for odd all odd numbers (then $x = \frac{x}{1}$), and never for even numbers (as any fraction equal to $\frac{x}{1}$ has an even numerator). Thus E_1 is the set of all odd integers. □

5a Write down the negation of 4

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, \forall z \in \mathbb{R} : z \geq y \Rightarrow x \leq 2z$$

in simplified form (the negation symbol itself is not allowed in your answer).
You only have to give your anwer, no explanation required.

Solution.

$$\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, \exists z \in \mathbb{R} : z \geq y \wedge x > 2z.$$ □

5b Prove or disprove the statement from 5a. 3

Solution. We prove the statement of 5a.

Let $x \in \mathbb{R}$ be arbitrary and choose $y = \frac{x}{2}$. Now let $z \in \mathbb{R}$ be arbitrary again. Suppose $z \geq y$, then $z \geq \frac{x}{2}$, so $x \leq 2z$. Thus the implication holds as desired. □

6a Show or disprove: $f : \mathbb{R} \to \mathbb{R}$ is increasing implies it is injective. 4

Note: f is increasing if $\forall x, y \in \mathbb{R} : x > y \Rightarrow f(x) > f(y)$.

Solution. We prove the statement is true.

Suppose f is increasing. Let $x, y \in \mathbb{R}$ be arbitrary. We prove $f(x) = f(y) \Rightarrow x = y$ using the contrapositive. Let $x \neq y$. Then either $x > y$ or $x < y$. In the first case, $x > y$, we have $f(x) > f(y)$, so $f(x) \neq f(y)$. The second case symmetrically shows $f(x) < f(y)$ and again $f(x) \neq f(y)$. In both cases $f(x) \neq f(y)$, so we have proven $x \neq y \Rightarrow f(x) \neq f(y)$ as desired.

Alternative: We can also use contrapositivity in the definition of increasing to see that increasing implies that $\forall x, y \in \mathbb{R} : f(x) \leq f(y) \Rightarrow x \leq y$. Now suppose f is increasing. Let $x, y \in \mathbb{R}$ be arbitrary. Suppose $f(x) = f(y)$. Then $f(x) \leq f(y)$ and $f(y) \leq f(x)$, so (by increasingness) both $x \leq y$ and $y \leq x$. Together this implies that $x = y$ as desired. □

6b Show or disprove: If $f : \mathbb{R} \to \mathbb{R}$ is injective, then it is either increasing or decreasing. 4

Solution. This is false. (Remark: For continuous functions it is true.)

A prove will (nearly) always use a counterexample. Make it as explicit as possible.

For example, take

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then if $f(x) = f(y)$ with $x, y \neq 0$ we have $\frac{1}{x} = \frac{1}{y}$, so $x = y$. Moreover if $f(x) = f(0) = 0$, then we must have $x = 0$ as well. Thus the function is injective.

Moreover $f(0) = 0 < f(2) = \frac{1}{2} < f(1) = 1$, so the function is neither increasing nor decreasing. □

Examiner resposible: Fokko van de Bult
Examination reviewer: Wolter Groenevelt