Midterm Mathematical Structures AM1010
Monday November 4, 2019, 9:00-11:00

TU Delft

No calculators allowed. Write the solutions in the fields provided. The grade is (score+6)/6.

1. (a) Write the statement below using quantifiers, logical operators, equations, and expressions such as $a \in \mathbb{R}$. 4

   There exists an integer x such that xy is an integer multiple of 7 for all rational numbers y.

   Don't use words or the divisor symbol $n \mid m$. No explanation necessary.

   Solution.
   $$\exists x \in \mathbb{Z} : \forall y \in \mathbb{Q} : \exists k \in \mathbb{Z} : xy = 7k.$$
   In this case it is a bit unclear where the for all y should be, so we also grade correct the following
   $$\forall y \in \mathbb{Q} : \exists x \in \mathbb{Z} : \exists k \in \mathbb{Z} : xy = 7k. \qquad \square$$
   However, whatever you do, you do need to introduce the y (using the quantifier) before you use it, so even though it is in the order of the text, it is incorrect to say

   INCORRECT: $$\exists x \in \mathbb{Z} : \exists k \in \mathbb{Z} : xy = 7k \ \forall y \in \mathbb{Q}$$

   (b) Write down the negation of the above statement (again using quantifiers, logical operators, equations, and expressions such as $a \in \mathbb{R}$). No explanation necessary. 3

   Solution.
   $$\forall x \in \mathbb{Z} : \exists y \in \mathbb{Q} : \forall k \in \mathbb{Z} : xy \neq 7k. \qquad \square$$

2. Let $f : A \to B$ be a function. Suppose $C \subseteq A$ and $D \subseteq B$. Show that 6
   $$f(C) \setminus D = f(C \setminus f^{-1}(D))$$
   using the definitions of image and pre-image.

   Solution. Let $x \in f(C) \setminus D$. Then $x \in f(C)$ and $x \notin D$. As $x \in f(C)$, there exists $y \in C$ such that $x = f(y)$. From $f(y) \notin D$ we obtain $y \notin f^{-1}(D)$. Thus $y \in C \setminus f^{-1}(D)$. It follows that $x = f(y) \in f(C \setminus f^{-1}(D))$. We conclude $f(C) \setminus D \subseteq f(C \setminus f^{-1}(D))$.

   Now let $x \in f(C \setminus f^{-1}(D))$. Then there exists $y \in C \setminus f^{-1}(D)$ such that $x = f(y)$. Thus $y \in C$ and $y \notin f^{-1}(D)$. As a consequence $x = f(y) \in f(C)$ and $x = f(y) \notin D$. We conclude $x \in f(C) \setminus D$. Thus $f(C \setminus f^{-1}(D)) \subseteq f(C) \setminus f^{-1}(D)$.

   From the two inclusions we derive $f(C) \setminus D = f(C \setminus f^{-1}(D))$. $\square$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define a relation R on $\mathbb{R}$ by saying xRy holds whenever $xf(y) = yf(x)$.

   (a) Is this relation reflexive (for any function f)? Prove or disprove. 2

   Solution. Yes. Let $x \in \mathbb{R}$. Then $xf(x) = xf(x)$, so xRx holds. □

   (b) Is this relation symmetric (for any function f)? Prove or disprove. 2

   Solution. Yes. Let $x, y \in \mathbb{R}$. Suppose xRy holds. Then $xf(y) = yf(x)$ and thus $yf(x) = xf(y)$. In particular yRx holds as well. □

   (c) Is this relation transitive (for any function f)? Prove or disprove. 3

   Solution. No. Take $f(x) = x^2$. Then $1R0$ holds as $1f(0) = 0 = 0f(1)$. $0R(-1)$ holds as $0f(-1) = 0 = -1 \cdot f(0)$. But $1R(-1)$ does not hold as $1f(-1) = 1 \neq -1 = (-1) \cdot f(1)$. □

   (d) Give an example of a function f for which this relation is an equivalence relation and show that this is the case. 2

   Solution. Take $f(x) = x$. Then xRy holds for all x and y, as $xf(y) = xy = yf(x)$. In particular this relation is transitive as well as symmetric and reflexive, so it is an equivalence relation.

   Remark: In fact almost any f works, but for not all functions is it this easy to conclude transitivity. □

4. Suppose $f : [0, 1] \to [0, 1]$ is increasing and $f(0) = 0$, $f(1) = 1$. Prove or disprove: f is surjective. 4

   Note: f is increasing if $\forall x, y \in \mathbb{R} : x > y \Rightarrow f(x) > f(y)$.

   Solution. The statement is not true. Take

$$f(x) = \begin{cases} \frac{x}{2} & x < \frac{1}{2} \\ \frac{x+1}{2} & x \geq \frac{1}{2}. \end{cases}$$

   Then $f(0) = \frac{0}{2} = 0$ and $f(1) = \frac{1+1}{2} = 1$. f is increasing as if $x < y < \frac{1}{2}$, then $f(x) = \frac{x}{2} < \frac{y}{2} = f(y)$. If $x < \frac{1}{2} \leq y$, then $f(x) = \frac{x}{2} < \frac{1}{4} < \frac{3}{4} \leq \frac{1+y}{2} = f(y)$. And if $\frac{1}{2} \leq x < y$, then $f(x) = \frac{x+1}{2} < \frac{y+1}{2} = f(y)$.

   Moreover $f(x) = \frac{1}{2}$ has no solutions, so f is not surjective. Indeed if $f(x) = \frac{1}{2}$ would have a solution $x < \frac{1}{2}$, then we would have $\frac{x}{2} = \frac{1}{2}$, so $x = 1 \geq \frac{1}{2}$, so this is impossible. Otherwise $x \geq \frac{1}{2}$ and we would have $\frac{x+1}{2} = \frac{1}{2}$, so $x = 0$, which is also not true.

   Remark: The statement is true for continuous functions, which follows from the intermediate value theorem, Theorem 5.3.6. □

5. In this exercise you only have to provide the final answer, no explanation necessary.

   (a) Write $\bigcap_{n\in\mathbb{N}}(2-\frac{1}{n}, 4-\frac{1}{n})$ as a finite union of intervals and finite sets. 2

   Solution. $[2,3)$.
   Note that the first set is $(1,3)$, so numbers ≥ 3 are missing in this set. Moreover, while 2 is contained in every set, if $x < 2$ there is an n such that $x < 2-\frac{1}{n}$ and $x \notin (2-\frac{1}{n}, 4-\frac{1}{n})$. □

   (b) Write $\bigcup_{n\in\mathbb{N}}(2-\frac{1}{n}, 4-\frac{1}{n})$ as a finite union of intervals and finite sets. 2

   Solution. $(1,4)$.
   Note that for $n = 1$ we have $(1,3)$, so numbers from 1 to 3 occur in that set. For numbers x with $3 \leq x < 4$ we have $x > 2-\frac{1}{n}$ for all n, and there is an n with $x < 4-\frac{1}{n}$, so $x \in (2-\frac{1}{n}, 4-\frac{1}{n})$. Also numbers ≥ 4 or < 1 do not appear in any of the sets. □

6. Prove using induction that $2n^3 - 3n^2 + n$ is a multiple of 6 for all $n \in \mathbb{N}$. 8

Solution. We prove this by induction.

For $n = 1$ we have $2 \cdot 1^3 - 3 \cdot 1^2 + 1 = 0$ is indeed a multiple of 6.

Suppose $2k^3 - 3k^2 + k = 6l$ for some integer l. Then

$$\begin{aligned} 2(k+1)^3 - 3(k+1)^2 + (k+1) &= 2k^3 + 6k^2 + 6k + 2 - 3k^2 - 6k - 3 + k + 1 \\ &= (2k^3 - 3k^2 + k) + 6k^2 = 6(l + k^2) \end{aligned}$$

Thus this is again a multiple of 6.

By induction we conclude that $2n^3 - 3n^2 + n$ is a multiple of 6 for all n. □

7. Complete the definition of $\lim s_n = s$ for a real number s. 2

The sequence (s_n) converges to s when ...

Solution. $\forall \epsilon > 0 : \exists N : \forall n > N : |s_n - s| < \epsilon$.

The book insists on $N \in \mathbb{N}$, and says $n \geq N$ instead of $n > N$, which is equivalent and you can do too. □

8. Prove using the definition of infinite limits that $\lim \frac{n^2+5}{n+3} = \infty$. 8

Solution. Let M be arbitrary. Choose $N = \max(2M, 3)$. Let $n > N$. Then $n + 3 < 2n$ as $n > 3$, so

$$\frac{n^2+5}{n+3} > \frac{n^2}{2n} = \frac{n}{2} > \frac{N}{2} \geq M.$$

□

9. Suppose (s_n) is a convergent sequence and (t_n) is divergent. Prove or disprove the following statements

   (a) $(s_n t_n)$ is divergent. 3

   Solution. This is false. Take $s_n = 0$ and $t_n = n$. Then $s_n \to 0$, whereas (t_n) diverges. $s_n t_n = 0$, so this converges to 0 as well.
   Remark: If $s = \lim s_n \neq 0$, then $(s_n t_n)$ diverges. □

   (b) $(s_n + t_n)$ is divergent. 3

   Solution. This is true, we prove it by contradiction. Suppose $(s_n + t_n)$ converges. As (s_n) converges, so does $(-s_n)$ (multiply by a constant). Thus $(t_n) = (s_n + t_n + (-s_n))$ converges as a sum of two convergent sequences. This is in contradiction with the fact that (t_n) diverges, so our assumption is false. This means that $(s_n + t_n)$ diverges. □

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