Test 1 Mathematical Structures AM1010
Solutions to online versions
Friday October 9, 2020, 13:45-14:35

The online exams consisted of mixes of the following problem. Each exam had one of the problems 1, one of the problems 2, etc. Below all solutions to all problems are given; you should look for the problem statement you had yourself.

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1 Problem 1

- Write down the negation of 4

$$\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : \forall z > 0 : x > y \wedge x > z \Rightarrow y < z$$

without using the symbol for negating a statement.

Solution.

$$\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : \exists z > 0 : x > y \wedge x > z \wedge y \geq z$$

□

- Write down the negation of the statement below without using the symbol for negating a statement. (No explanation necessary) 4

Schrijf de negatie van de uitspraak hieronder op zonder het symbool voor negatie te gebruiken. (Geen uitleg noodzakelijk)

$$\exists x \in \mathbb{R} : \exists y < 3 \in \mathbb{R} : \forall z \in \mathbb{R} : z \geq y \Rightarrow (x < y \vee y > z)$$

Solution.

$$\forall x \in \mathbb{R} : \forall y < 3 \in \mathbb{R} : \exists z \in \mathbb{R} : z \geq y \wedge (x \geq y \wedge y \leq z)$$

□

- Write down the negation of the statement below without using the symbol for negating a statement. (No explanation necessary) 4

Schrijf de negatie van de uitspraak hieronder op zonder het symbool voor negatie te gebruiken. (Geen uitleg noodzakelijk)

$$\exists x \geq 3 \in \mathbb{R} : \forall y \in \mathbb{R} : \forall z \in \mathbb{R} : (x > y \vee y < z) \Rightarrow z = y$$

Solution.

$$\forall x \geq 3 \in \mathbb{R} : \exists y \in \mathbb{R} : \exists z \in \mathbb{R} : (x > y \vee y < z) \wedge z \neq y$$

□

2 Problem 2

- Prove or disprove: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : xy \geq 0$. 4

  Solution. This is true.
  Take $x = 0$. Let y be arbitrary. Then $xy = 0 \geq 0$. □

- Prove or disprove: $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x < y^2$. 4

  Bewijs of ontkracht: $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x < y^2$.

  Solution. This is true. Let $x \in \mathbb{R}$ be arbitrary. Then take $y = \sqrt{|x|} + 1$, then $y^2 = |x| + 2\sqrt{|x|} + 1 > |x| \geq x$. □

- Prove or disprove: $\forall x \in \mathbb{Z} : \exists y \in \mathbb{Z} : x > y^2$. 4

  Bewijs of ontkracht: $\forall x \in \mathbb{Z} : \exists y \in \mathbb{Z} : x > y^2$.

  Solution. This is false. The negation is $\exists x \in \mathbb{Z} : \forall y \in \mathbb{Z} : x \leq y^2$. Now take $x = 0$, then indeed $y^2 \geq 0 = x$ for all y. (Note y is not complex.) □

3 Problem 3

- We define a relation R on $(0, \infty) \times (0, \infty)$ as $(x, y)R(u, v)$ iff $xv \geq yu$. This relation is reflexive.

  a Is the relation R symmetric? 4

  Solution. This is not true.
  Indeed $(2, 1)R(1, 2)$ as $4 \geq 1$, but not $(1, 2)R(2, 1)$ as $1 \geq 4$ does not hold. □

  b Is the relation R transitive? 4

  Solution. This is true.
  Let $(x, y), (u, v) and (a, b) \in (0, \infty)^2$ be arbitrary. Suppose $(x, y)R(u, v)$ and $(u, v)R(a, b)$. Then $xv \geq yu$ and $ub \geq va$. Thus $xv \cdot ub \geq yu \cdot va$ (all terms are positive). Dividing by uv (again $uv > 0$) we get $xb \geq ya$, so $(x, y)R(a, b)$ holds. □

  c Is the relation R an equivalence relation? If so, give a simple expression for the equivalence class $E_{(1,1)}$. 2

  Solution. No, it is not, as the relation is not symmetric. □

- We define a relation R on $\mathbb{Z} \times \mathbb{Z}$ as $(x, y)R(u, v)$ iff $xy \geq uv$. This relation is reflexive.

  De relatie R op $\mathbb{Z} \times \mathbb{Z}$ wordt gegeven door $(x, y)R(u, v)$ dan en slechts dan als $xy \geq uv$. Deze relatie is reflexief.

  a Is the relation R symmetric? Prove your result. 4

  Is de relatie R symmetrisch? Bewijs je antwoord.

Solution. The relation is not symmetric. Indeed $(2,2)R(1,1)$ holds as $2 \cdot 2 = 4 \geq 1 = 1 \cdot 1$, but $(1,1)R(2,2)$ does not hold as $1 \geq 4$ does not hold. □

b Is the relation R transitive? Prove your result. 4

Is de relatie R transitief? Bewijs je antwoord.

Solution. The relation is transitive. Indeed let $(x,y),(u,v),(w,z) \in \mathbb{Z}^2$. Suppose $(x,y)R(u,v)$ and $(u,v)R(w,z)$. Then both $xy \geq uv$ and $uv \geq wz$, so $xy \geq wz$. Therefore $(x,y)R(w,z)$ holds as well. □

c Is the relation R an equivalence relation? If so, give a simple expression for the equivalence class $E_{(1,1)}$. 2

Is de relatie R een equivalentierelatie? Zo ja, geef dan een eenvoudige uitdrukking voor de equivalentieklasse $E_{(1,1)}$.

Solution. As the relation is not symmetric, it is not an equivalence relation. □

- We define a relation R on $\mathbb{Q} \times \mathbb{Q}$ as $(x,y)R(u,v)$ iff $xv = yu$. This relation is reflexive.

  De relatie R op $\mathbb{Q} \times \mathbb{Q}$ wordt gegeven door $(x,y)R(u,v)$ dan en slechts dan als $xv > yu$. Deze relatie is reflexief.

  Sorry for the confusion: The English and Dutch versions do not correspond. We'll grade your answer correct whatever version you used. (The Dutch relation is also not reflexive, so that was not what was meant.)

a Is the relation R symmetric? Prove your result. 4

Is de relatie R symmetrisch? Bewijs je antwoord.

Solution. **English version:** R is symmetric. Indeed suppose $(x,y)R(u,v)$ holds. Then $xv = yu$, so $uy = vx$ and $(u,v)R(x,y)$ holds.

Nederlandse versie: R is niet symmetrisch: $(2,1)R(1,2)$ geldt omdat $2 \cdot 2 = 4 > 1 = 1 \cdot 1$, maar $(1,2)R(2,1)$ geldt niet want $1 \cdot 1 \geq 2 \cdot 2$ geldt niet. □

b Is the relation R transitive? Prove your result. 4

Is de relatie R transitief? Bewijs je antwoord.

Solution. **English version:** R is not transitive. Indeed $(1,1)R(0,0)$ as $1 \cdot 0 = 1 \cdot 0$, and $(0,0)R(2,1)$ as $0 \cdot 1 = 0 \cdot 2$, but $(1,1)R(2,1)$ does not hold as $1 \cdot 1 \neq 1 \cdot 2$.

Nederlandse versie: R is niet transitief. Er geldt $(1,1)R(-1,0)$ en $(-1,0)R(1,-1)$ want $1 \cdot 0 = 0 > -1 = 1 \cdot (-1)$ en $(-1) \cdot (-1) = 1 > 0 = 0 \cdot 1$, maar $(1,1)R(1,-1)$ geldt niet, want $1 \cdot (-1) \leq 1 \cdot 1$. □

c Is the relation R an equivalence relation? If so, give a simple expression for the equivalence class $E_{(1,1)}$. 2

Is de relatie R een equivalentierelatie? Zo ja, geef dan een eenvoudige uitdrukking voor de equivalentieklasse $E_{(1,1)}$.

Solution. **Both versions:** The relation R is not transitive, so R is not an equivalence relation. □

4 Problem 4

- a Show or disprove: If $f : [0,1] \to [3,5]$ is weakly increasing (that is $\forall x, y : x \geq y \Rightarrow f(x) \leq f(y)$), and $f(0) = 3$ and $f(1) = 5$, then f is surjective. 6

  Solution. This is not true.
  Indeed take
  $$f(x) = \begin{cases} 3 & x < 1 \\ 5 & x = 1 \end{cases}$$
  Then $f(0) = 3$, and $f(1) = 5$, and the function is weakly increasing. □

  b Show or disprove: Suppose $f : [0,1] \to [3,5]$ is weakly increasing. If f is surjective then $f(0) = 3$. 6

  Solution. This is true.
  If f is surjective, there must be an $x \in [0,1]$ such that $f(x) = 3$ as 3 is in the co-domain. But then $f(0) \leq f(x) = 3$. As $f(0) \in [3,5]$ also, we have $f(0) \geq 3$ as well. Together $3 \leq f(0) \leq 3$ implies $f(0) = 3$. □

- a Prove or disprove: If $f : A \to B$ is strictly decreasing (that is $\forall x, y : x > y \Rightarrow f(x) < f(y)$), and $g : B \to C$ is strictly increasing ($\forall x, y : x > y \Rightarrow f(x) > f(y)$), then $g \circ f$ is strictly decreasing. 6
  Bewijs of ontkracht: Als $f : A \to B$ strikt dalend is (dus $\forall x, y : x > y \Rightarrow f(x) < f(y)$), en $g : B \to C$ is strikt stijgend ($\forall x, y : x > y \Rightarrow f(x) > f(y)$), dan is $g \circ f$ strikt dalend.

  Solution. This is true. Suppose f is strictly decreasing and g is strictly increasing. Let x and y be arbitrary and suppose $x < y$. As f is strictly decreasing we have $f(x) > f(y)$. As g is strictly increasing this implies $g(f(x)) > g(f(y))$. Thus $g \circ f(x) > g \circ f(y)$ whenever $x < y$, so $g \circ f$ is strictly decreasing. □

  b Prove or disprove: Suppose $f : A \to B$ is strictly decreasing and $g : B \to C$ is such that $g \circ f$ is strictly decreasing. Then g is strictly increasing. 6
  Bewijs of ontkracht: Stel $f : A \to B$ is strikt dalend en $g : B \to C$ is zo dat $g \circ f$ strikt dalend is, dan is g strikt stijgend.

  Solution. This is false. For example take $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = -x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^2$. Then $g \circ f : [0, \infty) \to \mathbb{R}$ is given by $g \circ f(x) = g(-x) = -x^2$, is decreasing (watch the domain), but g itself is not. □

- a Prove or disprove: If $f : [0,1] \to [3,5]$ is weakly increasing (that is $\forall x, y : x \geq y \Rightarrow f(x) \geq f(y)$), and $f(0) = 3$ and $f(1) = 5$, then f is injective. 6
  Bewijs of ontkracht: Als $f : [0,1] \to [3,5]$ zwak stijgend is (dus $\forall x, y : x \geq y \Rightarrow f(x) \geq f(y)$), en $f(0) = 3$ en $f(1) = 5$, dan is f injectief.

  Solution. This is false. Indeed we can take
  $$f(x) = \begin{cases} 3 & x < 1 \\ 5 & x = 1 \end{cases}$$
  Then f is weakly increasing, $f(0) = 3$, $f(1) = 5$, and $f(1/2) = 3$ as well, so f is not injective. □

b Prove or disprove: Suppose $f : [0,1] \to [3,5]$ is weakly increasing. If f is injective then $f(0) \neq 5$. 6

Bewijs of ontkracht: Stel $f : [0,1] \to [3,5]$ is zwak stijgend. Als f injectief is, is $f(0) \neq 5$.

Solution. This is true. Indeed, suppose $f : [0,1] \to [3,5]$ is weakly increasing, injective and $f(0) = 5$. Then $f(1) \geq f(0) = 5$, but as $f(1) \in [3,5]$ (it has to be in the co-domain), we find $f(1) = 5$. But $f(1) = f(0)$ is in contradiction to the injectivity of f. Therefore $f(0) = 5$ cannot be true, and we find $f(0) \neq 5$. □

5 Problem 5

- Determine $\bigcap_{n\in\mathbb{Z}}[-2, n^2]$. Give a proof why your answer is correct! 6

  Solution. $\bigcap_{n\in\mathbb{Z}}[-2, n^2] = [-2, 0]$

  Indeed if $x \in \bigcap_{n\in\mathbb{Z}}[-2, n^2]$, then in particular $x \in [-2, 0^2] = [-2, 0]$. On the other hand if $x \in [-2, 0]$, then, as $n^2 \geq 0$, we have $x \in [-2, n^2]$ for all $n \in \mathbb{Z}$. Therefore $x \in \bigcap_{n\in\mathbb{Z}}[-2, n^2]$. □

- Determine $\bigcup_{x\in\mathbb{R}_{\geq 1}}[1 - 1/x, x)$. Give a proof why your answer is correct! 6

  Bepaal $\bigcup_{x\in\mathbb{R}_{\geq 1}}[1 - 1/x, x)$. Bewijs dat je antwoord klopt!

  You can assume / Je mag aannemen dat $1 - 1/x < x$ for / voor $x \in \mathbb{R}_{\geq 1}$

  Solution. $\bigcup_{x\in\mathbb{R}_{\geq 1}}[1 - 1/x, x) = [0, \infty)$.

  Indeed if $y \in \bigcup_{x\in\mathbb{R}_{\geq 1}}[1 - 1/x, x)$, then there is an x such that $y \in [1 - 1/x, x)$. As $1 - 1/x \geq 0$ for $x \geq 1$ we find $y \geq 0$ and so $y \in [0, \infty)$.

  On the other hand let $y \in [0, \infty)$, that is $y \geq 0$. Take $x = y + 1$. As $y^2 \geq 0$ we have $y(y+1) \geq y$, and $y + 1 > 0$ thus $y \geq \frac{y}{y+1} = 1 - \frac{1}{y+1} = 1 - \frac{1}{x}$. Also $y < y + 1 = x$. Therefore $y \in [1 - 1/x, x)$, and thus $y \in \bigcup_{x\in\mathbb{R}_{\geq 1}}[1 - 1/x, x)$. □

- Determine $\bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x]$. Give a proof why your answer is correct! You can assume $1 - 1/x < x$ for $x \in \mathbb{R}_{\geq 1}$ 6

  Bepaal $\bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x]$. Bewijs dat je antwoord klopt! Je mag aannemen dat $1 - 1/x < x$ for $x \in \mathbb{R}_{\geq 1}$

  You can assume / Je mag aannemen dat $1 - 1/x < x$ for / voor $x \in \mathbb{R}_{\geq 1}$

  Solution. We have $\bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x] = \{1\}$. Indeed $1 - 1/x < 1$ for all $x \geq 1$ and $1 \leq x$ for all $x \geq 1$, so $1 \in \bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x]$.

  On the other hand if $y \in \bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x]$, then $y \in (0, 1] = (1 - 1/1, 1]$, so $0 < y \leq 1$. Moreover if $0 < y < 1$, then we can take $x = \frac{1}{1-y}$ which satisfies $x \geq 1$ and we have $1 - 1/x = 1 - (1 - y) = y$, so $y \notin (1 - 1/x, x]$. Therefore $y \notin \bigcap_{x\in\mathbb{R}_{\geq 1}}(1 - 1/x, x]$. We conclude that only $y = 1$ can be in the intersection. □

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