Test 1 Mathematical Structures AM1010
Friday December 13, 2019, 9:00-10:00

No calculators allowed. Write the solutions in the fields provided. The grade is (score+4)/4.

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1 Let $S = \{\frac{n+2}{n} : n \in \mathbb{N}\}$. Determine $\inf(S)$ and $\sup(S)$. Prove your answer in detail. 5

Solution. $\inf(S) = 1$. Indeed $\frac{n+2}{n} = 1 + \frac{2}{n} > 1$ for all n, so 1 is a lower bound. Now suppose $\epsilon > 0$. Then, by the Archimedean property, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. Therefore $1 + \epsilon > 1 + \frac{2}{n}$ and $1 + \epsilon$ is not an lower bound to S. As any number greater than 1 is not a lower bound, whereas 1 is, we conclude that 1 is indeed the infimum of S.

Alternative: Notice that $\frac{n+2}{n}$ is a decreasing sequence with $\lim \frac{n+2}{n} = \lim 1 + \frac{2}{n} = 1$. Therefore all elements in the set S are bigger than 1, and numbers very close to 1 occur. Thus $\inf(S) = 1$.

$\sup(S) = 3$. Indeed for $n = 1$ we have $3 = \frac{1+2}{1} \in S$. Moreover as $n \geq 1$ for natural numbers, $\frac{n+2}{n} = 1 + \frac{2}{n} \leq 1 + \frac{2}{1} = 3$. We conclude that 3 is an upper bound which is in the set S, so 3 is a maximum. Therefore 3 is also the supremum of S. $\square$

2a Consider the sequence defined recursively as $a_{n+1} = \sqrt{3a_n + 1}$ and $a_1 = 1$. You may assume that the sequence is monotone. 9

Show that the sequence (a_n) converges. (if you use induction you need to write down a perfect induction proof).

Solution. We first show using induction that $0 < a_n < 8$ for all $n \in \mathbb{N}$. Indeed for $n = 1$ we have $0 < 1 = a_1 < 8$. Now suppose $0 < a_k < 8$. Then $a_{k+1} = \sqrt{3a_k + 1} > \sqrt{3 \cdot 0 + 1} = 1 > 0$. And $a_{k+1} = \sqrt{3a_k + 1} < \sqrt{3 \cdot 8 + 1} = 5 < 8$. Thus also $0 < a_{k+1} < 8$. By induction we find that $0 < a_n < 8$ for all $n \in \mathbb{N}$. In particular (a_n) is bounded.

By the monotone convergence theorem, the monotone, bounded sequence (a_n) thus converges. $\square$

2b Determine the limit $a = \lim a_n$ (for this part you can assume (a_n) converges). 4

Solution. Knowing that (a_n) converges we find that

$$a = \lim a_n = \lim a_{n+1} = \lim \sqrt{3a_n + 1} = \sqrt{3a + 1},$$

as Example 4.2.6 shows that $\lim \sqrt{t_n} = \sqrt{\lim t_n}$, which we can apply for $t_n = 3a_n + 1$.

Squaring this equation and rearranging gives $a^2 - 3a - 1 = 0$, so $a = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 + 4} = \frac{3}{2} \pm \frac{1}{2}\sqrt{13}$. As $a_n > 0$ for all n, we need to take the positive solution $a = \frac{3}{2} + \frac{1}{2}\sqrt{13}$. $\square$

3 Formulate the completeness axiom for the real numbers. 2

Solution. Any non-empty, bounded subset of the real numbers has a supremum. $\square$

4 What is the name of the axiom which says that $\forall x, y : x + y = y + x$? 2

Solution. Commutativity. □

5 Show that the sets $S = (0, \infty)$ are $T = [0, \infty)$ equinumerous by giving an explicit bijection. You don't have to show it is truly a bijection. 4

Solution. They are equinumerous. We define the following bijection $f : S \to T$

$$f(s) = \begin{cases} s-1 & s \in \mathbb{N} \\ s & s \notin \mathbb{N} \end{cases}$$

This function clearly maps integers to integers and non-integers to non-integers.

We now show the function is injective. If $f(s_1) = f(s_2)$, then either s_1 and s_2 are both integers or both non-integers. In the first case we have $s_1 = f(s_1) + 1 = f(s_2) + 1 = s_2$, in the second case $s_1 = f(s_1) = f(s_2) = s_2$. Either way, $s_1 = s_2$, thus the function is injective.

Now we show the function is surjective. Let $t \in T$. If t is an integer we have $t + 1 \in S$, and $f(t + 1) = t$. If t is not an integer, $t \neq 0$, so $t \in S$. And we have $f(t) = t$. Either way t is in the image of f, and f is surjective. □

6 Let (s_n) be a bounded sequence. Let M be such that $|s_n| < M$ for all n, and write $\limsup s_n = s$.

6a Give an example of a bounded sequence (s_n) such that $\limsup(s_n^2) \neq s^2$. Show this is the case by giving s_n, s and $\limsup(s_n^2)$. You don't have to give a proof that the lim sup's are what you say they are. 4

Solution. Take $s_n = (-1)^n - 1$, so $(s_n) = (0, -2, 0, -2, 0, -2, 0, -2, \ldots)$. Then $s = \limsup s_n = 0$, whereas $\limsup(s_n^2) = 4 \neq 0^2$. □

6b Suppose $s_n > 0$. Show that now $\limsup(s_n^2) = s^2$. 6

Solution. We show this using the characterization of Theorem 4.4.11.
Let $\epsilon > 0$. As $\limsup s_n = s$ there is N such that for all $n > N$ we have $s_n - s < \frac{\epsilon}{2M}$. Then let $n > N$ be arbitrary. Then

$$s_n^2 - s^2 = (s_n - s)(s_n + s) < \frac{\epsilon}{2M} \cdot (M + M) = \epsilon.$$

Here we use that $s \leq M$ if $s_n < M$ for all n. Also we use that $s_n + s > 0$ to ensure $(s_n - s)(s_n + s)$ can't become the product of two negative numbers, which makes the inequality valid.
Now for the second half, let $\epsilon > 0$ and let N be arbitrary. Then there is an $n > N$ such that $s_n > s - \frac{\epsilon}{2M}$. But then

$$s_n^2 > (s - \frac{\epsilon}{2M})^2 = s^2 - \frac{2s}{2M}\epsilon + \frac{\epsilon^2}{4M^2} > s^2 - \epsilon.$$

Here we again use that $s \leq M$. This argument fails if $s - \frac{\epsilon}{2M} < 0$, as then the inequality can not be squared. However when this happens we have $\epsilon > 2Ms > s^2$, so $s_n^2 > 0 > s^2 - \epsilon$ as well.

Alternatively for the second part:

$$s_n^2 - s^2 = (s_n - s)(s_n + s) > -\frac{\epsilon}{2M} \cdot (M + M) = -\epsilon.$$

In this case, as $-\frac{\epsilon}{2M} < 0$ and $M + M > 0$ the inequality works in this form.
Remark: Observe $M > |s_n| \geq 0$, so we don't divide our ϵ by 0. Choosing $s - \frac{\epsilon}{2s}$ in the second part does risk doing this.
Full Alternative Solution: We first show the following lemma

Lemma 1. *Let $S \subseteq [0, \infty)$ be a non-empty bounded set of positive numbers. Define $T = \{s^2 \ : \ s \in S\}$. Then* $\sup(T) = \sup(S)^2$.

Proof. Indeed suppose $t \in T$. Then there is an $s \in S$ with $t = s^2$. As $s \leq \sup(S)$ we find $t \leq \sup(S)^2$ (using that $s \geq 0$), so $\sup(S)^2$ is an upper bound to T.
Now let $x < \sup(S)^2$. If $x < 0$, then any $t \in T$ is bigger than x, so in that case x is not an upper bound. Otherwise $\sqrt{x}$ exists, and $\sqrt{x} < \sup(S)$ (using $\sup(S) \geq 0$). So there exists $s \in S$ with $s > \sqrt{x}$, and thus $s^2 > x$. As $s^2 \in T$ we conclude x is not an upper bound to T in this case either.
Thus we find that $\sup(S)^2$ is the lowest upper bound to T. □

Using the lemma we see that $\sup\{s_n^2 \ : \ n > N\} = (\sup\{s_n \ : \ n > N\})^2$, and therefore

$$\begin{aligned}\limsup s_n^2 &= \lim_{N\to\infty} \sup\{s_n^2 \ : \ n > N\} = \lim_{N\to\infty} (\sup\{s_n \ : \ n > N\})^2 \\ &= (\lim_{N\to\infty} \sup\{s_n \ : \ n > N\})^2 = (\limsup s_n)^2\end{aligned}$$

Remark 2: If you want to prove this using the definition of limit superior from the book you can't just use the following argument:
"For any convergent subsequence (s_{n_k}) we have $\lim(s_{n_k}^2) = (\lim s_{n_k})^2$, so the set S_2 of subsequential limits of (s_n^2) equals the set of squares of the set S_1 of subsequential limits of (s_n) itself: $S_2 = \{x^2 \ : \ x \in S_1\}$. "
Indeed, you would need to show that there are no divergent subsequences (s_{n_k}), for which $(s_{n_k}^2)$ does converge. If there were, one of these might have a higher limit than any square of an element in S_1. Considering this, an attempted proof along these lines is worth 0 points.
Note that such subsequences can exist if we don't insist on $s_n > 0$. Indeed if $s_n = (-1)^n$ the entire sequence (which is a subsequence of itself) diverges, but the sequence (s_n^2) does converge. □

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