Resit Exam Mathematical Structures TW1010
Thursday April 18, 2019, 9:00-12:00

No calculators allowed. Write the solutions in the fields provided. The grade is (score+8)/8.

1 Determine using a truth table whether or not $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology. 4

The truth table is given by

Solution.

p	q	$p \wedge q$	$\Rightarrow$	$p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	F

The statement is a tautology, since the column under $\Rightarrow$, determining the validity of the full statement only displays T's. □

2 Give a relation on $\mathbb{Q}$ which is transitive, reflexive, but not symmetric. 8

The relation R is defined as xRy holds whenever

Solution. $x \leq y$.

Of course there are many more examples, but this is a very well-known one. □

The relation R is reflexive as

Solution. For any $x \in \mathbb{Q}$ we have $x \leq x$, so xRx holds. □

The relation R is not symmetric as

Solution. Take $x = 1$ and $y = 2$. Then $1 \leq 2$, so $1R2$ holds, but not $2 \leq 1$, so $2R1$ does not hold. □

The relation R is transitive as

Solution. Suppose xRy and yRz hold, so both $x \leq y$ and $y \leq z$. Then $x \leq z$, thus xRz holds. □

3 Find the error in the following proof. 4

Theorem: For any function $f : \mathbb{R} \to \mathbb{R}$ we have $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

Proof:

1. Suppose $y \in f(A \setminus C)$.
2. Then there exists $x \in A \setminus C$ with $f(x) = y$.
3. Therefore $x \in A$ and $x \notin C$.
4. As $x \in A$ we have $f(x) \in f(A)$.
5. As $x \notin C$ we have $f(x) \notin f(C)$.
6. Hence $f(x) \in f(A) \setminus f(C)$.
7. As $y = f(x)$ we conclude $y \in f(A) \setminus f(C)$.
8. As we have shown for all y that $y \in f(A \setminus C) \Rightarrow y \in f(A) \setminus f(C)$ we have $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

The error in the proof occurs at line number $\boxed{5}$. This statement is wrong as

Solution. A counterexample is given by $f(x) = x^2$ and $C = \{2\}$, and $x = -2$. Then $x \notin C$, but $f(x) = 4 \in \{4\} = f(C)$.

Indeed this goes wrong for non-injective functions. □

4 Formulate the completeness axiom for the real numbers. 2

Solution. Any non-empty bounded subset of the real numbers has a supremum. □

5a Suppose $f : \mathbb{R} \to \mathbb{R}$ is a decreasing function and $A \subseteq \mathbb{R}$ is a bounded set. Show that 6

$$\sup(f(A)) \leq f(\inf(A)).$$

Solution. Take $y \in f(A)$, then there exists $a \in A$ with $f(a) = y$. As $a \geq \inf(A)$ and f is decreasing, we find that $y = f(a) \leq f(\inf(A))$. Therefore $f(\inf(A))$ is an upper bound to $f(A)$, and we conclude that $\sup(f(A)) \leq f(\inf(A))$. □

5b Give a decreasing function $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subseteq \mathbb{R}$ for which the strict inequality $\sup(f(A)) < f(\inf(A))$ holds; and show that your example works. 4

Solution. Take

$$f(x) = \begin{cases} 1 & x \leq 0, \\ 0 & x > 0. \end{cases}$$

and $A = (0, 1)$. Then $f(A) = \{0\}$, while $\inf(A) = 0$, so $f(\inf(A)) = 1$. In particular $\sup(f(A)) = \sup(\{0\}) = 0 \leq 1 = f(\inf(A))$.

As an example you can take any decreasing function with a jump at $\inf(A)$, for which the value at the jump is not the right-limit. □

6 Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, show that if $x, y \in \mathbb{R}$ with $x < y$, then there is a rational number $q \in \mathbb{Q}$ with $x < q < y$. 6

Solution. This is Theorem 3.3.13 from the book.

Let us first assume $x > 0$. By the Archimedean property there is an $n \in \mathbb{N}$ for which $\frac{1}{n} < y - x$. Consider the set $S = \{m \in \mathbb{N} \ : \ m > nx\}$, this is non-empty (by the Archimedean property again), so by wel-ordering it has a minimal element M. Then we find $x < \frac{M}{n}$ and $\frac{M-1}{n} < x$ (as otherwise either $M - 1 \in S$ contradicting that M is minimal, or $M = 1$ and we would have $x \leq 0$ contradicting $x > 0$). But then $y = x + (y - x) > \frac{M-1}{n} + \frac{1}{n} = \frac{M}{n}$. Thus $q = \frac{M}{n}$ satisfies the conditions of the statement.

In the case $x \leq 0$ there is an positive integer k with $k > -x$, so $0 < x + k < y + k$, and we find that there is a $q' \in \mathbb{Q}$ with $x + k < q' < y + k$. But then $q' - k \in \mathbb{Q}$ satisfies $x < q < y$ as desired. □

7 The sequence (a_n) is defined recursively as $a_{n+1} = \sqrt{8 + \frac{1}{2}a_n a_{n-1}}$, starting with $a_1 = 1$ and $a_2 = 2$.

7a Use induction to prove that (a_n) is increasing. 7
Hint: Use the statement $P(n) : a_n \leq a_{n+1} \leq a_{n+2}$.

Solution. We will prove by induction that $a_n \leq a_{n+1} \leq a_{n+2}$ for all $n \in \mathbb{N}$.

We calculate $a_3 = \sqrt{8 + \frac{1}{2} \cdot 1 \cdot 2} = \sqrt{9} = 3$, so $a_1 = 1 \leq a_2 = 2 \leq a_3 = 3$. Therefore the base condition holds.

Observe that $a_n \geq 0$ for all n, as this is true for $n = 1, 2$ and for $n \geq 3$ the a_n is defined as the (positive) square root of some number.

Now assume $a_k \leq a_{k+1} \leq a_{k+2}$ for some k. Then we have

$$a_{k+3} = \sqrt{8 + \frac{1}{2}a_{k+1}a_{k+2}} \geq \sqrt{8 + \frac{1}{2}a_{k+1}a_k} = a_{k+2}$$

where we use that $a_{k+2} \geq a_k$ and that $a_{k+1} \geq 0$. As the induction hypothesis already gives $a_{k+1} \leq a_{k+2}$ we can conclude that $a_{k+1} \leq a_{k+2} \leq a_{k+3}$ holds.

By induction we see that for all n we have $a_n \leq a_{n+1} \leq a_{n+2}$ and thus that the sequence increases.

Note that you can alternatively set up an induction proof following exercise 3.1.27. □

7b We still use the sequence (a_n) defined by $a_{n+1} = \sqrt{8 + \frac{1}{2}a_n a_{n-1}}$, $a_1 = 1$, and $a_2 = 2$.

Show that (a_n) converges. 6

Solution. We show that the sequence is bounded above by 10 using induction. The induction hypothesis is $a_n \leq 10 \wedge a_{n+1} \leq 10$.

Indeed $a_1 \leq 10$ and $a_2 \leq 10$. Moreover if for some k we have $a_k, a_{k+1} \leq 10$, then

$$a_{k+2} = \sqrt{8 + \frac{1}{2}a_k a_{k+1}} \leq \sqrt{8 + \frac{1}{2} \cdot 10 \cdot 10} = \sqrt{58} \leq 10.$$

(Note that we again use that $a_k, a_{k+1} \geq 0$.) By induction we find that the sequence is bounded by 10.

By the monotone convergence theorem we can now conclude that this increasing bounded sequence converges. □

7c Determine the limit $\lim a_n = a$. 2

Solution. We have

$$a = \lim a_{n+1} = \lim \sqrt{8 + \frac{1}{2}a_n a_{n-1}} = \sqrt{8 + \frac{1}{2}a \cdot a} = \sqrt{8 + \frac{1}{2}a^2}.$$

Here we use the rules of calculations for limits (including that we can move a limit through a square root, conform Example 4.2.6). Squaring this equation gives $a^2 = 8 + \frac{1}{2}a^2$, so $\frac{1}{2}a^2 = 8$, so $a^2 = 16$ and $a = \pm 4$. As we already noted that $a_n \geq 0$ for all n, the limit must be positive as well, and $a = 4$ is the only viable option. □

8 Suppose $f(x)$ satisfies $|f(x)-f(y)| \leq \sqrt{|x-y|}$. Suppose (s_n) is a Cauchy sequence, show that $(f(s_n))$ is also a Cauchy sequence using the definition of Cauchy sequence. 6

Solution. Let $\epsilon > 0$. Then there is an N such that for all $n, m > N$ we have $|s_n - s_m| < \epsilon^2$. Then for this same N we have for all $n, m > N$ that

$$|f(s_n) - f(s_m)| \leq \sqrt{|s_n - s_m|} < \sqrt{\epsilon^2} = \epsilon,$$

thus the sequence $(f(s_n))$ is also Cauchy. □

The axioms of an ordered field as applied to $\mathbb{R}$ are

- A1 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x + y = w + z$;
- A2 $\forall x, y \in \mathbb{R} : x + y = y + x$;
- A3 $\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$;
- A4 $\exists 0 : \forall x \in \mathbb{R} : x + 0 = x$ and this 0 is unique;
- A5 $\forall x \in \mathbb{R} : \exists(-x) \in \mathbb{R} : x + (-x) = 0$ and $(-x)$ is unique;
- M1 $\forall x, y \in \mathbb{R} : x \cdot y \in \mathbb{R}$ and $x = w \wedge y = z \Rightarrow x \cdot y = w \cdot z$;
- M2 $\forall x, y \in \mathbb{R} : x \cdot y = y \cdot x$;
- M3 $\forall x, y, z \in \mathbb{R} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- M4 $\exists 1 \neq 0 : \forall x \in \mathbb{R} : x \cdot 1 = x$ and this 1 is unique;
- M5 $\forall x \neq 0 : \exists(1/x) \in \mathbb{R} : x \cdot (1/x) = 1$ and $(1/x)$ is unique;
- DL $\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = x \cdot y + x \cdot z$;
- Oj Omitted from the solutions as irrelevant;

9 Show using the axioms that $(x + y)^2 = x^2 + (2(xy) + y^2)$. 6
Here we use the notations $x^2 = x \cdot x$ and $2 = 1 + 1$.
Be sure to precisely indicate what axioms you use in each step.

Solution. It is a bit of a tedious calculation; the point of which is to be really careful you don't take illegal shortcuts.

$$\begin{aligned}
(x+y)(x+y) &\overset{\text{DL}}{=} (x+y)x + (x+y)y \overset{\text{M2}}{=} x(x+y) + y(x+y) \\
&\overset{\text{DL}}{=} (x^2 + xy) + (yx + y^2) \overset{\text{A3}}{=} x^2 + (xy + (yx + y^2)) \\
&\overset{\text{A3}}{=} x^2 + ((xy + yx) + y^2) \overset{\text{M2}}{=} x^2 + ((xy + xy) + y^2) \\
&\overset{\text{M4}}{=} x^2 + (((xy) \cdot 1 + (xy) \cdot 1) + y^2) \overset{\text{DL}}{=} x^2 + ((xy)(1+1) + y^2) \\
&\overset{\text{M2}}{=} x^2 + (2(xy) + y^2).
\end{aligned}$$

□

10 Give the definition of convergence of a series. A series $\sum_{n=1}^{\infty} a_n$ converges if 2

Solution. The sequence (s_k) of partial sums converges. Here $s_k = \sum_{n=1}^{k} a_n$. □

11 Determine whether or not the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges, and if it does converge, whether this convergence is absolute or conditional. 4

Solution. First we consider absolute convergence, that is the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$. As this is just a shifted p-series with $p = \frac{1}{2} \leq 1$, this series diverges.

Now the series itself is alternating, and the absolute values of the terms $(\frac{1}{\sqrt{n+1}})$ form a decreasing sequence that converges to 0, so by the alternating series test the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

As the series converges, but the sum of the absolute values diverges we conclude that this series is conditionally convergent. □

12 Determine for all x whether $\sum_{n=1}^{\infty} \frac{(2n)!}{n!} x^n$ converges or diverges. Also determine when the series is absolutely or conditionally convergent. 5

(Fill in things like $x \in [2, 3)$ or $x = 5$ in the boxes below after doing your calculations.)

- The series converges absolutely for: $x = 0$
- The series converges conditionally for: never
- The series diverges for: $x \neq 0$

Solution. We use the ratio test to calculate the radius of convergence. Indeed we have

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{(2n)!}{n!} \frac{(n+1)!}{(2n+2)!} = \lim \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!}$$
$$= \lim \frac{n+1}{(2n+1)(2n+2)} = \lim \frac{1}{4n+2} = 0.$$

With a radius of convergence of 0, the series only converges at the center $x = 0$, and this convergence is always absolute. □

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