Faculty of Electrical Engineering, Mathematics and Computer Science
Numerical Methods I, TW2060, BSc Applied Mathematics
Exam, June 28, 2019, 13:30 - 16:30

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$$\text{Grade of exam} = \frac{\text{Round}(\frac{2\Sigma}{3})}{2}, \quad \text{where } \Sigma = \text{ total sum over all subquestions.}$$

NB. This exam contains three questions!

1 We consider two versions of Taylor's Theorem, applied to linearisation of a function $f(x)$ around $x = a$, in which two different formulas for the remainder (that is the error) are considered. The first version of Taylor's Theorem reads as:

Theorem 1.1: Suppose that $a < b \leq x < c$, and let $f \in C^2(a,c)$ (that is f has continuous second order derivatives on (a,x)), then there is a $\xi \in (a,x)$ such that

$$f(x) = f(b) + (x-b)f'(b) + \frac{1}{2}(x-b)^2 f''(\xi). \tag{1}$$

The first focus is on Theorem 1.1. We prove Theorem 1.1 by assuming that the error behaves quadratically, that is, we assert

$$f(x) - (f(b) + (x-b)f'(b)) = A(x-b)^2, \tag{2}$$

for a certain $A \in \mathbb{R}$.

a We introduce the function $\phi(t) = f(t) - (f(b) + (t-b)f'(b)) - A(t-b)^2$, where all the conditions in Theorem 1.1 are satisfied. Show that $\phi(b) = 0 = \phi(x)$ and that $\phi'(b) = 0$. (2pt)

b Prove that there is a $\xi \in (b,x)$ such that $A = \frac{1}{2}f''(\xi)$, and complete the proof of Theorem 1.1. (2pt)

Secondly, we focus on Theorem 1.2, which reads as

Theorem 1.2: Suppose that $b < x$, and let $f, f', f'' \in L^2(b,x)$ (that is f, f' and f'' are square integrable over (b,x)), then

$$f(x) = f(b) + (x-b)f'(b) + \int_b^x (x-t)f''(t)dt. \tag{3}$$

c Prove Theorem 1.2 (in particular equation (3). *Hint: Use integration by parts.* (2pt)

d Suppose that there an $M > 0$ such that $|f''(t)| \leq M$ for $t \in [b,x]$. Prove that both versions of Taylor's Theorem (Theorem 1.1 and Theorem 1.2) imply the following upper bound for the error:

$$|f(x) - (f(b) + (x-b)f'(b))| \leq \frac{1}{2}M(x-b)^2. \tag{4}$$

(2pt)

2 We consider the following boundary value problem (convection-diffusion equation):

$$\begin{cases} -\varepsilon y'' + y' = 0, & \text{for } x \in (0,1), \\ y(0) = 0, & y(1) = 1, \end{cases} \tag{5}$$

where $\varepsilon > 0$.

a Motivate why we can expect problems as $\varepsilon \longrightarrow 0$ in boundary value problem (5). (1pt) b Prove that $y(x) = \frac{\exp(\frac{x}{\varepsilon})-1}{\exp(\frac{1}{\varepsilon})-1}$ is a solution to boundary value problem (5). (1pt)

c We use the finite difference method to approximate the solution to boundary value problem (5). We use central differences to approximate the first-order term and a mesh with nodal points $x_j = j\Delta x$, where $x_{n+1} = 1$. Derive the set of linear equations for general n. Write the system in the form of $A\mathbf{w} = \mathbf{b}$ (where $\mathbf{w}$ represents the approximation), give the entries of A and of $\mathbf{b}$. Take care of the boundary conditions. (3pt)

d Show that the local truncation error, defined by $\varepsilon = A\mathbf{y} - \mathbf{b}$, where $\mathbf{y}$ represents the exact solution, is of order $\mathcal{O}(\Delta x^2)$ as $\Delta x \to 0$. (2pt)

e We are going to analyse the behaviour of the numerical approximation with respect to spurious oscillations. For the numerical approximation, we set the following power relation $w_j = r^j$, where $r \in \mathbb{R}$. Show that this gives the following quadratic equation for r

$$-\frac{\varepsilon}{\Delta x^2}(r^2 - 2r + 1) + \frac{1}{2\Delta x}(r^2 - 1) = 0, \tag{6}$$

with solutions $r = 1$ and $r = \frac{\frac{\Delta x}{2\varepsilon}+1}{1-\frac{\Delta x}{2\varepsilon}}$ for $1 - \frac{\Delta x}{2\varepsilon} \neq 0$. (2pt)

f Show that we have to choose $\Delta x < 2\varepsilon$ in order to prevent spurious oscillations. (2pt)

3 We consider the generic initial value problem

$$y' = f(t, y(t)), \qquad y(t_0) = y_0, \tag{7}$$

of which we approximate the solution by the following predictor-corrector method

$$\begin{cases} w_* = w_n + \frac{\Delta t}{2} f(t_n, w_n), \\ \\ w_{n+1} = w_n + \Delta t f(t_n + \frac{1}{2}\Delta t, w^*). \end{cases} \tag{8}$$

a Show that the local truncation error is of order $\mathcal{O}(\Delta t^2)$. It is not allowed to use the test equation. (3pt)

b We consider the following system of differential equations

$$\begin{aligned} y_1' &= -2y_1 + y_2, \\ y_2' &= y_1 - 2y_2, \end{aligned} \tag{9}$$

which values of Δt give a stable numerical numerical integration to the above system if we use method (8)? (2pt)

c Does the numerical approximation obtained by method (8) applied to problem (9) converge? Motivate your answer. (2pt)

d As the initial condition, we use $y_1(0) = 1$, and $y_2(0) = 1$, use method (8) to approximate the solution of problem (9) after one time-step of $\Delta t = 0.5$. (2pt)

e Compare the behaviour of numerical stability of method (8) to the stability of Heun's method, which reads as

$$\begin{cases} w_* = w_n + \Delta t f(t_n, w_n), \\ \\ w_{n+1} = w_n + \frac{\Delta t}{2}(f(t_n, w_n) + f(t_n + \Delta t, w_*)). \end{cases} \tag{10}$$

(2pt)