Faculty of Electrical Engineering, Mathematics and Computer Science
Numerical Methods I, TW2060, BSc Applied Mathematics
Exam, June 28, 2019, 13:30 - 16:30, ANSWERS

Responsible examiner: F.J. Vermolen
Exam reviewer: C. Vuik.

ANSWERS

1 a Let

$$\phi(t) = f(t) - (f(b) + (t-b)f'(b)) - A(t-b)^2, \tag{1}$$

then

$$\phi(b) = f(b) - (f(b) + (b-b)f'(b)) - A(b-b)^2 = f(b) - f(b) = 0.$$

Further it was given (see equation (2) in the exam) that

$$f(x) - (f(b) + (x-b)f'(b)) = A(x-b)^2, \tag{2}$$

which implies

$$f(x) - (f(b) + (x-b)f'(b)) - A(x-b)^2 = 0. \tag{3}$$

Substituting $t = x$ into equation (1), gives

$$\phi(x) = f(x) - (f(b) + (x-b)f'(b)) - A(x-b)^2 = 0,$$

where the last equality follows from equation (3). Hence $\phi(b) = 0$ and $\phi(x) = 0$. From the definition of $\phi(t)$, it follows that

$$\phi'(t) = f'(t) - (f'(b) - 2A(t-b)) \Longrightarrow \phi'(b) = f'(b) - (f'(b) - 2A(b-b)) = 0.$$

b Note that $\phi \in C^2(a,c)$ and $a < b < x < c$. Since $\phi(b) = \phi(x) = 0$, it immediately follows from the Mean Value Theorem that $\exists \eta \in (b,x)$ such that $\phi'(\eta) = 0$. Note that we already concluded in part a that $\phi'(b) = 0$, and note that $\phi \in C^2(a,c)$ and $a < b < x < c$, then we use the Mean Value Theorem again to conclude that $\exists \xi \in (b,\eta) \subset (b,x)$ such that $\phi''(\xi) = 0$.
The second derivative of ϕ is found easily:

$$\phi''(t) = f''(t) - 2A.$$

Since $\phi''(\xi) = 0$, it follows that $f''(\xi) = 2A \Longrightarrow A = \frac{f''(\xi)}{2}$. From equation (2), it follows that

$$f(x) = f(b) + (x-b)f'(b) + \frac{f''(\xi)}{2}(x-b)^2,$$

which concludes the proof of Theorem 1.1.

c We perform integration by parts on the last term on the right-hand side in equation (3) of the exam, to get

$$\int_b^x (x-t)f''(t)dt = \left[(x-t)f'(t)\right]_b^x + \int_b^x f'(t)dt = -(x-b)f'(b) + f(x) - f(b).$$

In the last step, the Main Theorem from Integration has been used. This implies that

$$f(x) = f(b) + (x-b)f'(b) + \int_b^x (x-t)f''(t)dt,$$

which concludes the proof of Theorem 1.2.

d Suppose that there an $M > 0$ such that $|f''(t)| \leq M$ for $t \in [b,x]$. First, we focus on the bound from Theorem 1.1. Then from Theorem 1.1, it follows that there is a $\xi \in (b,x)$ such that

$$f(x) - (f(b) + (x-b)f'(b)) = \frac{1}{2}(x-b)^2 f''(\xi)$$

$$\Longrightarrow |f(x) - (f(b) + (x-b)f'(b))| = \frac{1}{2}(x-b)^2 |f''(\xi)| \leq \frac{1}{2}M(x-b)^2.$$

The last inequality follows from the given hypothesis that there an $M > 0$ such that $|f''(\xi)| \leq M$ for $\xi \in [b,x]$.
Secondly, we focus on Theorem 1.2. Theorem 1.2 implies

$$f(x) - (f(b) + (x-b)f'(b)) = \int_b^x (x-t) f''(t) dt$$

$$\Longrightarrow |f(x) - (f(b) + (x-b)f'(b))| = |\int_b^x (x-t) f''(t) dt| \leq M \int_b^x (x-t) dt = \frac{M}{2}(x-b)^2.$$

The second last step follows from the hypothesis $|f''(t)| \leq M$ for $t \in (b,x)$ (and from Theorem 1.6.5 (The Mean Value Theorem for Integration) in the book).

2 We consider the following boundary value problem (convection-diffusion equation):

$$\begin{cases} -\varepsilon y'' + y' = 0, & \text{for } x \in (0,1), \\ y(0) = 0, & y(1) = 1, \end{cases} \tag{4}$$

where $\varepsilon > 0$.

a If $\varepsilon = 0$, then the problem becomes

$$y' = 0, \text{ for } x \in (0,1),$$

$$y(0) = 0, \ y(1) = 1,$$

Since $y' = 0$, this implies that $y = C$, where C is a constant. From $x = 0$, we have $C = 0$, however, from $x = 1$, we get $C = 1$. Hence we end up at a contradiction.

b We can use multiple ways to demonstrate that $y(x) = \frac{\exp(\frac{x}{\varepsilon})-1}{\exp(\frac{1}{\varepsilon})-1}$ is a solution to the boundary value problem. We will use substitution, differentiation gives

$$y'(x) = \frac{\exp(\frac{x}{\varepsilon})}{\varepsilon(\exp(\frac{1}{\varepsilon})-1)} \Longrightarrow y''(x) = \frac{\exp(\frac{x}{\varepsilon})}{\varepsilon^2(\exp(\frac{1}{\varepsilon})-1)}.$$

Substitution into the differential equation, gives

$$-\varepsilon y'' + y' = -\varepsilon \frac{\exp(\frac{x}{\varepsilon})}{\varepsilon^2(\exp(\frac{1}{\varepsilon})-1)} + \frac{\exp(\frac{x}{\varepsilon})}{\varepsilon(\exp(\frac{1}{\varepsilon})-1)} = 0.$$

Hence the expression for $y(x)$ satisfies the differential equation. Subsequently, the boundary conditions have to be checked. That is $y(0) = \frac{\exp(\frac{0}{\varepsilon})-1}{\exp(\frac{1}{\varepsilon})-1} = 0$ and $y(1) = \frac{\exp(\frac{1}{\varepsilon})-1}{\exp(\frac{1}{\varepsilon})-1} = 1$. Hence $y(x)$ satisfies the boundary value problem.

c First, we write the derivatives at points x_j in terms of finite differences, and we use the notation $y(x_j) = y_j$, which gives

$$y''(x_j) = \frac{y_{i+1} - 2y_j + y_{j-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2),$$

$$y'(x_j) = \frac{y_{j+1} - y_{j-1}}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

Substitution into the differential equation gives

$$0 = -\varepsilon y'' + y' = -\varepsilon \frac{y_{i+1} - 2y_j + y_{j-1}}{\Delta x^2} + \frac{y_{j+1} - y_{j-1}}{2\Delta x} + \mathscr{O}(\Delta x^2).$$

Next, we neglect the error, and use w_j to denote the numerical approximation of y_j, to obtain the following (linear) algebraic equation for w_j:

$$-\varepsilon \frac{w_{i+1} - 2w_j + w_{j-1}}{\Delta x^2} + \frac{w_{j+1} - w_{j-1}}{2\Delta x} = 0, \qquad \text{for } j \in \{1, \ldots, n\}. \tag{5}$$

As boundary conditions, we have $y_0 = 0$, and $y_{n+1} = 1$ (since $x_0 = 0$ and $x_{n+1} = 1$), this translates into $w_0 = 0$ and $w_{n+1} = 1$, this gives for $j = 1$:

$$-\varepsilon \frac{w_2 - 2w_1}{\Delta x^2} + \frac{w_2}{2\Delta x} = 0. \tag{6}$$

For $j = n$, we get

$$-\varepsilon \frac{-2w_n + w_{n-1}}{\Delta x^2} - \frac{w_{n-1}}{2\Delta x} = \frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} \tag{7}$$

Equation (5) represents the j-th equation of the system of linear equations $A\mathbf{w} = \mathbf{b}$, hence the entries of A (a_{ij}) become

$$a_{j\ j-1} = -\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x},\ a_{j\ j} = \frac{2\varepsilon}{\Delta x^2},\ a_{j\ j+1} = -\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x},\ b_j = 0,\ j \in \{2, \ldots, n-1\}.$$

From equation (6), we get for the first row

$$a_{11} = \frac{2\varepsilon}{\Delta x^2},\ a_{12} = -\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x},\ b_1 = 0.$$

For the last row, we get

$$a_{n\ n-1} = -\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x},\ a_{n\ n} = \frac{2\varepsilon}{\Delta x^2},\ b_n = \frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x}.$$

The discretisation matrix A is given by

$$A = \begin{pmatrix} \frac{2\varepsilon}{\Delta x^2} & -\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x} & 0 & \ldots & 0 & \\ -\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} & \frac{2\varepsilon}{\Delta x^2} & -\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & -\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} & \frac{2\varepsilon}{\Delta x^2} & -\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x} \\ 0 & \ldots & \ldots & \ldots 0 & -\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} & \frac{2\varepsilon}{\Delta x^2} \end{pmatrix},$$

and the right-hand side vector $\mathbf{b}$ is given by

$$\mathbf{b} = \begin{pmatrix} 0 \\ \ldots \\ \ldots \\ 0 \\ \frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} \end{pmatrix}$$

d We use the definition. First, we consider the 1-st row:

$$\varepsilon_1 = (A\mathbf{y})_1 - b_1 = -\varepsilon \frac{y_2 - 2y_1}{\Delta x^2} + \frac{y_2}{2\Delta x} - 0 = -\varepsilon \frac{y_2 - 2y_1 + y_0}{\Delta x^2} + \frac{y_2 - y_0}{2\Delta x} - (-\varepsilon y''(x_1) + y'(x_1))$$

The last equation follows from using the boundary condition $y(0) = y(x_0) = 0$ and $-\varepsilon y''(x_1) + y'(x_1) = 0$. Next we use Taylor expansions for $y(x_2)$ and $y(x_0)$ around x_1:

$$y_0 = y(x_0) = y(x_1 - \Delta x) = y_1 - \Delta x\, y'(x_1) + \frac{\Delta x^2}{2} y''(x_1) - \frac{\Delta x^3}{3!} y'''(x_1) + \mathscr{O}(\Delta x^4)$$

$$y_2 = y(x_2) = y(x_1 + \Delta x) = y_1 + \Delta x\, y'(x_1) + \frac{\Delta x^2}{2} y''(x_1) + \frac{\Delta x^3}{3!} y'''(x_1) + \mathscr{O}(\Delta x^4)$$

Substitution of the expressions of y_0 and y_2 into the expression for ε_1, gives

$$\varepsilon_1 = \mathcal{O}(\Delta x^2).$$

For $j \in \{2, \ldots, n-1\}$, it follows

$$\varepsilon_j = (A\mathbf{y})_j - b_j = -\varepsilon \frac{y_{i+1} - 2y_j + y_{j-1}}{\Delta x^2} + \frac{y_{j+1} - y_{j-1}}{2\Delta x} - 0$$

$$-\varepsilon \frac{y_{i+1} - 2y_j + y_{j-1}}{\Delta x^2} + \frac{y_{j+1} - y_{j-1}}{2\Delta x} - (-\varepsilon y''(x_j) + y'(x_j))$$

Again we use Taylor expansions for y_{j-1} and y_{j+1} around x_j:

$$y_{j-1} = y(x_{j-1}) = y(x_j - \Delta x) = y_j - \Delta x\, y'(x_j) + \frac{\Delta x^2}{2} y''(x_j) - \frac{\Delta x^3}{3!} y'''(x_j) + \mathcal{O}(\Delta x^4)$$

$$y_{j+1} = y(x_{j+1}) = y(x_j + \Delta x) = y_j + \Delta x\, y'(x_j) + \frac{\Delta x^2}{2} y''(x_j) + \frac{\Delta x^3}{3!} y'''(x_j) + \mathcal{O}(\Delta x^4),$$

to arrive at

$$\varepsilon_j = (A\mathbf{y})_j - b_j = -\varepsilon \frac{y_{i+1} - 2y_j + y_{j-1}}{\Delta x^2} + \frac{y_{j+1} - y_{j-1}}{2\Delta x} - (-\varepsilon y''(x_j) + y'(x_j)) = \mathcal{O}(\Delta x^2).$$

The same procedure (Taylor series) is applied to ε_n with $b_n = \frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x} = \varepsilon \frac{y_{n+1}}{\Delta x^2} - \frac{y_{n+1}}{2\Delta x}$ (using $y_{n+1} = 1$), to arrive at

$$\varepsilon_n = (A\mathbf{y})_n - b_n = -\varepsilon \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} + \frac{y_{n+1} - y_{n-1}}{2\Delta x} - (-\varepsilon y''(x_n) + y'(x_n)) = \mathcal{O}(\Delta x^2)$$

From this analysis, it can be concluded that the local truncation error is of order $\mathcal{O}(\Delta x)^2$ as $\Delta x \to 0$.

e We set $w_j = r^j$, hence $w_{j-1} = r^{j-1}$ and $w_{j+1} = r^{j+1}$, hence $w_{j+1} = r w_j = r^2 w_j$. Substitution into the finite difference equation (5), gives

$$-\varepsilon \frac{r^{j+1} - 2r^j + r^{j-1}}{\Delta x^2} + \frac{r^{j+1} - r^{j-1}}{2\Delta x} = 0.$$

Division by r^{j-1} ($r = 0$ is a non-interesting trivial solution), gives

$$-\varepsilon \frac{r^2 - 2r + 1}{\Delta x^2} + \frac{r^2 - 1}{2\Delta x} = 0, \text{ for } r \neq 0,$$

which is what we were supposed to derive. Subsequently, using $r^2 - 2r + 1 = (r-1)^2$ and $r^2 - 1 = (r+1)(r-1)$, immediately reveals that $r = 1$ and $r = \frac{\frac{\Delta x}{2\varepsilon} + 1}{1 - \frac{\Delta x}{2\varepsilon}}$, if $1 - \frac{\Delta x}{2\varepsilon} \neq 0$.

f The exact solution, see assignment 2b, rules out any oscillatory behaviour. The solution consists of two non-trivial modes: $r = 1$ and $r = \frac{\frac{\Delta x}{2\varepsilon} + 1}{1 - \frac{\Delta x}{2\varepsilon}}$. Since $w_j = r^j$, the first solution ($r = 1$) is a constant mode. The second mode varies over j, it permits oscillatory behaviour if $r < 0$, hence in order to prevent any oscillatory behaviour, we have to require that $r > 0$. Since $\varepsilon > 0$ and $\Delta x > 0$, the numerator is always positive, whereas the denominator is given by $1 - \frac{\Delta x}{2\varepsilon}$, which is positive if and only if $1 - \frac{\Delta x}{2\varepsilon} > 0$, hence we require that $\Delta x < 2\varepsilon$ to prevent any spurious oscillatory behaviour.

3 We consider the generic initial value problem

$$y' = f(t, y(t)), \qquad y(t_0) = y_0, \tag{8}$$

of which we approximate the solution by the following predictor-corrector method

$$\begin{cases} w_* = w_n + \frac{\Delta t}{2} f(t_n, w_n), \\ \\ w_{n+1} = w_n + \Delta t f(t_n + \frac{1}{2}\Delta t, w_*). \end{cases} \tag{9}$$

a We use the definition of the local truncation error at time-step t_{n+1}

$$\tau_{n+1}(\Delta t) := \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{10}$$

where y_{n+1} represents the (exact) solution to the initial value problem and z_{n+1} represents the numerical approximation using the (exact) solution $y_n := y(t_n)$ from the previous time-step. The solution at t_{n+1} is expressed by the use of a Taylor series about t_n:

$$y_{n+1} = y(t_n + \Delta t) = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \frac{\Delta t^3}{3!} y'''(\xi),$$

for a $\xi \in (t_n, t_{n+1})$ from Taylor's Theorem. Next, we express z_{n+1} from y_n:

$$\begin{aligned} & z_* = y_n + \frac{\Delta t}{2} f(t_n, y_n), \\ & z_{n+1} = y_n + \Delta t f(t_n + \frac{\Delta t}{2}, z_*) = y_n + \Delta t f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t_n, y_n)) = \\ & y_n + \Delta t (f(t_n, y_n) + \frac{\Delta t}{2} \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\Delta t}{2} f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y} + \mathscr{O}(\Delta t^2)). \end{aligned} \tag{11}$$

The latest steps follow from application of Taylor's Theorem for multivariate functions. Since $y' = f(t, y(t))$, we have

$$y''(t) = \frac{d}{dt} f(t, y(t)) = \frac{\partial f(t, y(t))}{\partial t} + \frac{\partial f(t, y(t))}{\partial y} y'(t) = \frac{\partial f(t, y(t))}{\partial t} + \frac{\partial f(t, y(t))}{\partial y} f(t, y(t)).$$

This implies that the second equation in (11) can be written as

$$z_{n+1} = y_n + \Delta t (y'(t_n) + \frac{\Delta t}{2} y''(t_n) + \mathscr{O}(\Delta t^2)) = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathscr{O}(\Delta t^3).$$

Hence equation (10) becomes

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t} = \frac{y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \frac{\Delta t^3}{3!} y'''(\xi) - (y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathscr{O}(\Delta t^3))}{\Delta t} = \mathscr{O}(\Delta t^2).$$

This is exactly what we were supposed to prove.

b In order to assess stability of the numerical method, we first derive the amplification factor by the use of the test equation $y' = \lambda y$. Let w_n be the numerical approximation of the (exact) solution y_n, then

$$\begin{aligned} & w_* = w_n + \frac{\Delta t}{2} \lambda w_n, \\ & w_{n+1} = w_n + \Delta t \lambda w_* = w_n + \Delta t \lambda (w_n + \frac{\Delta t}{2} \lambda w_n) = w_n (1 + \lambda \Delta t + \frac{\lambda^2 \Delta t^2}{2}). \end{aligned}$$

Since $Q(\lambda \Delta t) := \frac{w_{n+1}}{w_n}$, it follows directly that

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{\lambda^2 \Delta t^2}{2}.$$

Subsequently, we compute the eigenvalues of the matrix, A, in our system of equations, where it can be seen easily that we can write the system in the form

$$\mathbf{y}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{y} = A\mathbf{y}$$

The eigenvalues of A are computed by setting

$$\det(A - \lambda I) = 0,$$

which give $\lambda_1 = -1$ and $\lambda_2 = -3$, hence $\lambda(A) = \{-1, -3\}$. Stability of the numerical solution is obtained by requiring

$$|Q(\lambda \Delta t)| \leq 1.$$

In the current case, the eigenvalues are real, this implies $Q(\lambda \Delta t) \in \mathbb{R}$, and that we obtain

$$-1 \leq 1 + \lambda \Delta t + \frac{\lambda^2 \Delta t^2}{2} \leq 1.$$

This implies that

$$(I): \; 0 \leq 2 + \lambda \Delta t + \frac{\lambda^2 \Delta t^2}{2}$$

$$(II): \; \lambda \Delta t + \frac{\lambda^2 \Delta t^2}{2} \leq 0.$$

Note that $\lambda \Delta t = 0$ implies that inequality (I) is satisfied, further, note that the discriminant is always negative, which implies that there are no real-valued roots in case of equality. Hence inequality (I) is always satisfied.

The second inequality is processed further. Note that $\lambda \Delta t$ can be factored out, and note that $\lambda \Delta t < 0$, this gives

$$1 + \frac{\lambda \Delta t}{2} \geq 0 \Longleftrightarrow 1 - \frac{|\lambda| \Delta t}{2} \Longleftrightarrow \Delta t \leq \frac{2}{|\lambda|}.$$

Since $\lambda_1 = -1$ and $\lambda_2 = -3$ are the eigenvalues of A, the larger eigenvalue in absolute value is the more restrictive one, hence stability of the numerical solution is warranted if

$$\Delta t \leq \frac{2}{3}.$$

c In part a of this assignment, we proved that the local truncation error converges to zero as the time-step is sent to zero, in other words, we have

$$\tau_{n+1}(\Delta t) \longrightarrow 0 \text{ as } \Delta t \longrightarrow 0.$$

Therefore, according to the definition of consistency, the method that we currently consider is consistent.

We further observed that the method is stable if $\Delta t \leq \frac{2}{3}$, hence sending Δt to zero makes the method stable.

Lax' Theorem says that a consistent, stable method converges. Therefore, as Δt is sent to zero, the method converges, which amounts to the fact that the glocal truncation error

$$\varepsilon_{n+1}(\Delta t) = y_{n+1} - w_{n+1} \longrightarrow 0 \text{ as } \Delta t \longrightarrow 0.$$

Therewith, the answer is affirmative.

d The use of $y_1(0) = 1$, and $y_2(0) = 1$, implies that the predictor $\mathbf{w}_*$ is given by

$$\mathbf{w}_* = \mathbf{w}_0 + \frac{\Delta t}{2} A\mathbf{w}_0 = \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix}-2 & 1\\1 & -2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{4}\begin{pmatrix}1\\1\end{pmatrix} = \frac{3}{4}\begin{pmatrix}1\\1\end{pmatrix}.$$

Subsequently, for the corrector $\mathbf{w}_1$, we get

$$\mathbf{w}_1 = \mathbf{w}_0 + \Delta t A\mathbf{w}_* = \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-2 & 1\\1 & -2\end{pmatrix}\cdot\frac{3}{4}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{8}\begin{pmatrix}-2 & 1\\1 & -2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{8}\begin{pmatrix}1\\1\end{pmatrix} = \frac{5}{8}\cdot\begin{pmatrix}1\\1\end{pmatrix}.$$

e The amplification factor of Heun's (Modified Euler) method is given by

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \frac{\lambda^2\Delta t^2}{2}.$$

Hence, this amplification is identical to the amplification method that we derived in assignment 3b. Therefore, the stability properties of Heun's method and the method that we analysed earlier are identical (hence application to the problem that we considered earlier, gives $\Delta t \leq \frac{2}{3}$ again).