Technische Universiteit Delft
Fac. Elektrotechniek, Wiskunde en Informatica

Examination Valuation of Derivatives, Wi 3405TU

Friday January 27nd 2016, 9:00 - 11:00 **(2 hours examination)**

1. Consider the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with a payoff function as the final condition at $t = T$.

a. Show that the transformations $S = e^y$, $\tau = T - t$, and $v(y, \tau) = e^{r\tau}V(y, \tau)$, followed by $x = y + (r - \frac{1}{2}\sigma^2)\tau$ result in the following heat equation for unknown $u(x, \tau)$,

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2},$$

Answer: One starts with the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

supplemented with the terminal and boundary conditions. The change of independent variables $S = e^y$, $t = T - \tau$, results in

$$S\frac{\partial}{\partial S} \to \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau},$$

$$\frac{\partial V}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial y} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2} + rV = 0.$$

If we replace $v(y, \tau) = e^{r\tau}V(y, \tau)$, we will obtain:

$$\frac{\partial v}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial y} - \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2} = 0,$$

The substitution $x = y + (r - \sigma/2)\tau$ allows us to eliminate the first-order term, and to reduce the preceding equation to the form with dependent variable $u(x, \tau)$:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2},$$

b. Write down the Forward-in-Time Central-in-Space scheme, FTCS, for this heat equation. Show that the discrete scheme is second-order accurate in space and first-order accurate in time.
Answer: The FTCS scheme for this heat equation is exactly as (23.7) on page 241, except a multiplication by $\sigma^2/2$. The accuracy is discussed and derived, based on the Taylor expansion, on pages 246, 247 in Equations (23.13), (23.14).

c. Apply von Neumann stability analysis for this heat equation. Show that the stability condition takes the form $\sigma^2 \Delta\tau \leq (\Delta x)^2$.

Answer: This is the derivation on page 248, which needs to be done properly by the student, and the result follows from the inclusion of the term $\sigma^2/2$.

2. A double barrier option has two barriers, B_1 and B_2, one above and one below the current stock price, $B_1 < S_0 < B_2 < E$. In a *one-touch double barrier put option*, the option is knocked out, resulting in a valueless option, if at least one of the barriers is breached during the life of the option.

We assume that the asset follows the process that leads to the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

a. Why would an investor buy such an option, and why would a writer sell such an option?
Answer: A writer would sell such an option because it may be relatively easy to hedge it, and the possible losses in the case of an unhedged position may be limited as the pay-off is strictly bounded.

An investor would buy such an option if he/she has very specific views about the possible size of the asset movements, and thinks that the movement stays with the $[B_1, B_2]$ range. With a somewhat limited pay-off the option is cheaper than a usual put option. However, it can be mentioned that also a usual put option has a limited payoff. The current one is however cheaper.

b. Transform the equations by $\tau = T - t$ into forward equations in time. Give the appropriate computational domain, the corresponding boundary conditions and the payoff function for the one-touch double barrier put option.
Answer: The computational domain is from B_1 to B_2. On both boundaries the barrier put option values equals 0. The payoff is the put payoff, provided that the asset value stays within the range $[B_1, B_2]$. This payoff is now an initial condition, as we have to transform the PDE to an equation forward in time.

c. Which standard (single) barrier options should be in a portfolio with this double barrier option in order to replicate a regular European put option?
Answer: With the notion, in the case of single barrier options, barrier "in + out = European option", we need, in the case of a double barrier option, a "down-and-in" barrier put option, with barrier B_1, and an "up-and-in" barrier option, with barrier level B_2 to guarantee that also in the double barrier case, we end up with a European put option payoff.

d. Why does it not make much sense to have $B < E$, with E the strike price, in an up-and-in barrier call option?
Answer: When $B < E$ in an up-and-in barrier call option, the situation would just resemble a regular call option. The option gets in the money once the stock price passes barrier B, but it has to move further ($S > E$) to end up in the money. So, it would just be equal to a common call option, and therefore it would be silly to have a barrier there.

3. The following Matlab code is given:

```
%%%%%%%%%% Problem and method parameters %%%%%%%%%%%%
S = 3; E = 2; T = 1; r = 0.05; sigma = 0.3;
M = 400; dt = T/M; p =0.5;
u = exp(sigma*sqrt(dt) + (r-0.5*sigma^2)*dt);
d = exp(-sigma*sqrt(dt) + (r-0.5*sigma^2)*dt);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Time T option values
W = max(E-S*d.^([M:-1:0]').*u.^([0:M]'),0);

% Work back to option value at time zero
for i = M:-1:1
    W = exp(-r*dt)*(p*W(2:i+1) + (1-p)*W(1:i));
end

disp('Option value is'), disp(W)
```

– What is the computational technique in this Matlab code and what is computed? Adapt the Matlab code above to value a Bermudan put option with two early-exercise dates, at $t = 0.5$ and at $T = 1$.
Answer: This is the computation by means of the binomial tree model of a European put option., with 400 time steps.

If we need to compute a Bermudan option, with exercise at $t = 0.5$ and at $T = 1$, we'd need to add a statement that when we reach t=0.5 (time step 200) we have to choose the maximum of continuation value, which is now ialready computed in the European case, and the payoff function. So, we'd exercise when the payoff value is higher than the continuation value. We need not change the code for intermediate time steps, when we just need to compute accurate continuation values. Also at final time, exercise is automatically taken into account here.

Place your name and study number on each page with solutions.