Exam Option Valuation Methods
wi3405tu
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All questions have equal weight

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1 A *forward contract* operates as follows. At time $t = 0$, Party A agrees to purchase an asset from Party B at a specified delivery time T for a specified price F. (Note that Party A is committed to the future purchase – by contrast, with a European call option the holder has the right, but not the obligation, to buy at the prescribed price.)

A Show that $V(S,t) = S_0 e^{rt} - S$ is a solution to the Black-Scholes equation and argue why this shows that $F = S_0 e^{rT}$ is a fair price.

C Determine the replicating portfolio at times $t = 0$ and $t = T$ of a forward contract with price $F = S_0 e^{rT}$.

2 The Black-Scholes price of a European Call is $C(S,t) = SN(d_1) - Ee^{r(T-t)}N(d_2)$ where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and N is the normal distribution. See equations 8.19 and 8.20 in your book.

A Since d_1 depends on S and t it is more accurate to write $d_1(S,t)$. Determine $\lim_{t\uparrow T} N(d_1(S,t))$.

B Prove that $N(d_1(S,t))$ solves the Black-Scholes equation. You may use that $\frac{\partial C}{\partial S} = N(d_1)$ according to equation 9.1 in your book.

C A *one-zero* option pays 1 euro if $S(T) < E$ and pays nothing if $S(T) > E$. Determine the Black-Scholes value of a one-zero option at time $t = 0$.

3 A Using equation 10.5, $\frac{\partial C}{\partial t} = \frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) - rEe^{-r(T-t)}N(d_2)$, prove that $\frac{\partial C}{\partial t} \leq 0$ and give a financial argument to support this.

B Use put-call parity to derive a similar equation in the case of a European Put $\frac{\partial P}{\partial t}$.

C Determine $\lim_{S\downarrow 0} \frac{\partial P}{\partial t}$. Use your result to argue that this Greek may be positive or negative.

4 A Let $P(S, E, T)$ be the price of a put with expiry T, strike E, and underlying asset S. Prove that

$$P(S, E, T_2) > P(S, E, T_1)$$

if $T_2 > T_1$ and the interest rate r is **negative**.

B Consider the code of ch07.m, which plots fifty asset paths:

```
%CH07   Program for Chapter 7
%
% Plot discrete sample paths

randn('state',100)
clf

%%%%%%%%%  Problem parameters  %%%%%%%%%%%
S = 1; mu = 0.05; sigma = 0.5; L = 1e2; T = 1; dt = T/L; M = 50;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

tvals = [0:dt:T];
Svals = S*cumprod(exp((mu-0.5*sigma^2)*dt + sigma*sqrt(dt)*randn(M,L)),2);
Svals = [S*ones(M,1) Svals];  % add initial asset price
plot(tvals,Svals)
title('50 asset paths')
xlabel('t'), ylabel('S(t)')
```

I add two lines to the code:

$$\mathtt{Payoff} = \mathtt{max}(1 - \mathtt{Svals}(:,\mathtt{L}), 0); \mathtt{Value} = \mathtt{exp}(-\mathtt{mu} * \mathtt{T}) * \mathtt{mean}(\mathtt{Payoff})$$

What have I computed? Be precise and motivate your answer!