Delft University of Technology **Faculty EEMCS**

Exam Ordinary Differential Equations, AM2030
Monday 27 January 2020, 13.30-16.30h

- This exam consists of 5 problems.
- All answers need to be justified.
- Norm: total of 37 points; the distribution of points is as shown in the exercises. The exam grade is (total points +3)/4.

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1. The following differential equation for the function $x : \mathbb{R} \to \mathbb{R}$ is given

$$t^2 \frac{d^2x}{dt^2} + t\frac{dx}{dt} + x = 1 + t, \qquad t > 0.$$

a. (2) Find the homogeneous solution $x_h(t)$.

b. (2) Find a particular solution $x_p(t)$, by trying a polynomial.

c. (2) Determine the solution with initial condition $x(1) = 1, x'(1) = 1$.

2. (6) Compute a fundamental matrix solution for the system

$$\dot{\mathbf{x}} = A\mathbf{x},$$

with A given by

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix},$$

and give the general solution and its interval of existence.

3. Consider the following system of differential equations for $(x(t), y(t))$:

$$\begin{aligned} \frac{dx}{dt} &= -y + xf(x,y) \\ \frac{dy}{dt} &= \ \ x + yf(x,y), \end{aligned} \tag{1}$$

with

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = 0 \end{cases}$$

a. (1) Calculate the equilibrium point(s).

b. (4) Linearize your system about the origin O and determine the stability (stable, asymptotically stable, unstable) of O and classify (saddle,center, spiral/focus, (critical) node). What does the stability of O in the linearized system tell you about the stability of O in the nonlinear system (1) ?

c. (2) Rewrite system (1) in polar coordinates $(r(t), \theta(t))$ and show that this leads to the following system

$$\frac{dr}{dt} = r^3 \sin(\pi/r),$$
$$\frac{d\theta}{dt} = 1. \tag{2}$$

d. (4) Calculate the limit cycles (periodic solutions) and sketch the phase portrait.

4. (5) Use the Laplace transform to solve the initial value problem

$$y'' + y = \begin{cases} t^2, & 0 < t < 1 \\ 0, & 1 \leq t < \infty \end{cases},$$

with $y(0) = y'(0) = 0$.

5. The following differential equation for $\mathbf{y}(t)$ is given:

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y}, \tag{3}$$

where $A(t)$ is an $n \times n$ matrix that depends on time. Assume that $\mathbf{x}(t)$ is a known solution and that we try to find the remaining solutions $\mathbf{y}(t)$ of the form

$$\mathbf{y}(t) = \phi(t)\mathbf{x}(t) + \mathbf{z}(t), \tag{4}$$

with $\mathbf{z}(t) = [0, z_2, \cdots, z_n]^T$ an n-dimensional vector in $\mathbb{R}^n$ and $\phi(t)$ a C^1 function of t.

a. (2) Prove that $\mathbf{z}(t)$ has to satisfy

$$\dot{\mathbf{z}} = A\mathbf{z} - \dot{\phi}\mathbf{x},$$

in order for $\mathbf{y}(t)$ to be a solution of (3)

b. (2) Demonstrate that the i-th component of $\mathbf{z}(t) = z_i(t)$, satisfies

$$\dot{z}_i = \sum_{j=2}^{n} A_{ij} z_j - \dot{\phi} x_i, \qquad i = 1, 2, \ldots, n$$

c. (2) Next we assume that $x_1(t) \neq 0$. Show that

$$\dot{z}_i = \sum_{j=2}^{n} \left(A_{ij} - \frac{x_i}{x_1} A_{1j} \right) z_j,$$

and

$$\phi(t) = \int \frac{1}{x_1(t)} \sum_{j=2}^{n} A_{1j}(t) z_j(t)\, dt$$

and hence a solution $\mathbf{y}(t)$ of the form assumed is obtained.

d. (3) Now take $n = 2$ and show that $\mathbf{x}(t)$ and the solution $\mathbf{y}(t) = \phi(t)\mathbf{x}(t) + \mathbf{z}(t)$, are linearly independent solutions.

Hint: You may you still assume that $x_1(t) \neq 0$.