Exercise 1:

a) instationary heat flow, u = temp., source Q, heat flow through boundary at $x=0$ is zero (insulated), heat flow at $x=L$ is proportional to the temp at $x=L$, $f(x)$ is initial temp.
1pt.

b) equilibrium sol.:
$$\begin{cases} u_{xx}+1=0 \\ u_x(0)=0 \\ u_x(L)=-Au(L) \end{cases} \Rightarrow \begin{array}{l} u(x)=-\frac{1}{2}x^2+c_1x+c_0 \\ u_x(0)=0 \Rightarrow c_1=0 \\ u_x(L)=-Au(L) \Leftrightarrow -L+c_1=-A\cdot(-\frac{1}{2}L^2+c_1L+c_0) \end{array}$$

$$\Rightarrow \quad c_1=0$$

$$-L-\frac{1}{2}AL^2=-Ac_0 \Rightarrow c_0=\frac{L+\frac{1}{2}AL^2}{A} \text{ 1½pt} \quad \text{for } (A\neq 0) \overset{A>0}{\Rightarrow} u(x)=-\frac{1}{2}x^2+\frac{L+\frac{1}{2}AL^2}{A}$$

if $A=0$ then $-L=0$ $\Rightarrow$ no solution. ½pt

c) separation of variables
$$\begin{cases} \frac{T'}{T}=\frac{X''}{X}=-\lambda \\ X'(0)=X'(L)=0 \end{cases} \overset{\lambda\geq 0}{\Rightarrow} \begin{array}{ll} X(x)=c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x), & \lambda>0 \\ X(x)=c_1x+c_2, & \lambda=0 \end{array}$$

$$\Rightarrow \dots \Rightarrow \lambda_n=\left(\frac{n\pi}{L}\right)^2, \quad n=0,1,2,\dots, \text{ and } X_n(x)=\cos\left(\frac{n\pi x}{L}\right) \quad \text{1pt}$$

$$\left.\begin{array}{l} u(x,t)=\sum_{n=0}^{\infty} a_n e^{-\lambda_n t}\cos(\sqrt{\lambda_n}x) \\ u(x,0)=f(x) \end{array}\right\} \Rightarrow a_n=\frac{\langle f(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} \quad \text{1pt}$$

d) eigenfunction expansion; boundary conditions should (be made) homogeous or not they are hom. (Green's 2nd id.)
1pt
↓
subst. into PDE, use orthogonality
adjust to .., ..

1pt

e) Assume there are 2 different solutions u_1 and u_2. Consider $v=u_1-u_2$, then

$$\begin{cases} v_t=v_{xx} \\ v_x(0,t)=0 \\ v_x(L,t)=-Av(L,t) \\ v(x,0)=0 \end{cases} \text{1pt} \quad *v, \int_0^L \dots dx \Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^L v^2dx=\left[vv_x\right]_{x=0}^{L}-\int_0^L v_x^2dx$$

$$=-Av^2(L,t)-\int_0^L v_x^2dx \leq 0 \quad \text{1pt}$$

$$\Rightarrow \int_0^L v^2(x,t)dx \leq \int_0^L v^2(x,0)dx=0 \overset{v\in C}{\Rightarrow} v(x,t)\equiv 0 \Rightarrow u_1\equiv u_2$$

there can not be two different solutions $\Rightarrow$ there is at most one solution (uniqueness) 1pt

EXERCISE 2:

a) vibrating string, displ. u, initial displ. f, initial velocity g

BC at $x=0$: force proportional to displ (spring), velocity (damping), external force ($h(t)$)

1pt

b) BC $u_x(0,t) = -h(t)$; $u(x,t) = F(x-t) + G(x+t)$

IC $\begin{cases} u(x,0) = 0, & x>0 \\ u_t(x,0) = 0, & x>0 \end{cases} \Rightarrow \begin{array}{l} F(x) + G(x) = 0, \; x>0 \\ -F'(x) + G'(x) = 0, \; x>0 \end{array} \Rightarrow G(x) = F(x) + k_1 \Rightarrow$

$\begin{cases} F(x) = -\frac{k_1}{2} & \text{for } x>0 \\ G(x) = \frac{k_1}{2} & \text{for } x>0 \end{cases}$ 1pt

$y(t) = F(-t) \Rightarrow \frac{dy(t)}{dt} = F'(-t) \cdot -1$

$u_x(0,t) = -h(t), t\geq 0 \Rightarrow F'(-t) + \underbrace{G'(t)}_{=0} = -h(t) \Rightarrow \frac{dy}{dt}(t) = h(t) \Rightarrow y(t) = y(0) + \int_0^t h(s)ds$

$\Rightarrow F(-t) = F(0) + \int_0^t h(s)\,ds$ for $t\geq 0$ 1pt (continuity $F \Rightarrow F(0) = -\frac{k_1}{2}$

So, $u(x,t) = \begin{cases} 0 & \text{for } x-t>0 \\ \int_0^{t-x} h(s)\,ds & \text{for } x-t<0 \end{cases}$ 1pt

c) BC $u(0,t) = 0 \Rightarrow$ odd extension of all functions (in $x=0$) w.r.t. $x=0$ 1pt

$\mathcal{F}(u(x,t)) = U(\omega,t) \Rightarrow \begin{cases} U_{tt} = -\omega^2 U \\ U(\omega,0) = F(\omega), U_t(\omega,0) = 0 \end{cases} \Rightarrow U(\omega,t) = F(\omega)\cos(\omega t)$ 1pt

$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} F(\omega) \frac{e^{i\omega t} + e^{-i\omega t}}{2} e^{-i\omega x} d\omega = \frac{1}{2}\int_{-\infty}^{\infty}\left(F(\omega)e^{-i\omega(x-t)} + F(\omega)e^{-i\omega(x+t)}\right)d\omega$

$= \frac{1}{2}f(x-t) + \frac{1}{2}f(x+t)$, where f is odd w.r.t. its argument zero. 1pt

d) $\begin{cases} u_{tt} - u_{xx} = 0 \\ u_x(0,t) = -h(t) \\ u(x,0) = 0, \; u_t(x,0) = 0 \end{cases}$ $\mathcal{L}(u(x,t)) = U(x,s) \Rightarrow s^2U(x,s) - s\overbrace{u(x,0)}^{f(x)=0} - \overbrace{u_t(x,0)}^{=0} - U_{xx}(x,s) = 0$

$\Rightarrow \begin{cases} s^2U(x,s) - U_{xx}(x,s) = 0 \\ U_x(0,s) = -H(s) \end{cases}$

$U(x,s) = A(s)e^{-sx} + \underbrace{B(s)e^{sx}}_{\text{unbounded} \Rightarrow B(s)\equiv 0} \Rightarrow U_x(0,s) = -sA(s)e^{-s0} = -H(s) \Rightarrow A(s) = \frac{H(s)}{s}$ 1pt

$U(x,s) = \frac{H(s)}{s}e^{-sx}$ 1pt $\Rightarrow U_x(x,s) = -H(s)\cdot e^{-sx} \Rightarrow u_x(x,t) = -\int_0^t h(t-\tilde{t})\,\delta(\tilde{t}-x)\,d\tilde{t} = -H(t-x)\,h(t-x)$

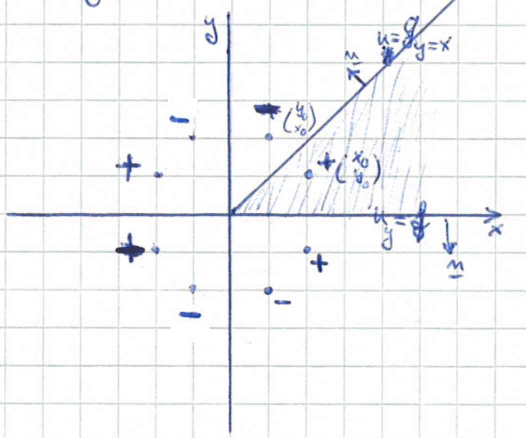
$\Rightarrow u(x,t) = \underbrace{u(0,t)}_{\substack{=0 \\ \text{involved} \\ \text{in ICs} \\ \text{and PDE}}} - \int_0^x H(t-\tilde{x})\,h(t-\tilde{x})\,d\tilde{x} = \begin{cases} 0 & \text{for } t-x<0 \\ \int_0^{t-x} h(s)\,ds & \text{for } t-x>0 \end{cases}$ 1pt

Exercise 3 :

a) stationary heat flow in domain (Q), heatflow through $y=0$ given by f ; temp. at $y=x$ is given by g

1pt

b)



$$n=\begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$$

$1\frac{1}{2}$ pt

$$G(\underline{x};\underline{x}_0)=\frac{1}{2\pi}\ln\left|\underline{x}-\binom{x_0}{y_0}\right|-\frac{1}{2\pi}\ln\left|\underline{x}-\binom{y_0}{x_0}\right|+\frac{1}{2\pi}\ln\left|\underline{x}-\binom{x_0}{-y_0}\right|-\frac{1}{2\pi}\ln\left|\underline{x}-\binom{y_0}{-x_0}\right|$$
$$-\frac{1}{2\pi}\ln\left|\underline{x}-\binom{-y_0}{x_0}\right|+\frac{1}{2\pi}\ln\left|\underline{x}-\binom{-x_0}{y_0}\right|+\frac{1}{2\pi}\ln\left|\underline{x}-\binom{-x_0}{-y_0}\right|-\frac{1}{2\pi}\ln\left|\underline{x}-\binom{-y_0}{-x_0}\right|$$

$1\frac{1}{2}$ pt

c)
$$\int_D(\underbrace{u\,\Delta G}_{\delta(\underline{x}-\underline{x}_0)}-\underbrace{G\,\Delta u}_{Q(x,y)})\,dV=\int_{\partial D}\left(u\frac{\partial G}{\partial n}-G\frac{\partial u}{\partial n}\right)dS=\int_0^\infty \underbrace{\frac{\partial u}{\partial y}}_{f(x)}G\Big|_{y=0}dx+\int_\infty^0 g\underbrace{\frac{\partial G}{\partial n}}_{\frac{1}{\sqrt{2}}\left(-\frac{\partial G}{\partial x}+\frac{\partial G}{\partial y}\right)\big|_{y=x}}\Big|_{y=x}dx$$

1pt 1pt

$$\Rightarrow\quad u(\underline{x}_0)=\int_D G.Q\,dV+\int_0^\infty f(x).\,G\Big|_{y=0}dx\pm\int_0^\infty g(x)\frac{\partial G}{\partial n}\Big|_{y=x}dx\qquad \text{with } \frac{\partial G}{\partial n}\Big|_{y=x}=\binom{G_x}{G_y}\binom{-1/\sqrt{2}}{1/\sqrt{2}}$$

$\int_D = \int_0^\infty\int_0^x \dots\,dy\,dx$

1pt

d) Assume: there are two different sol. u_1 and u_2 : Consider $v=u_1-u_2$, then

$$\begin{cases} v_{xx}+v_{yy}=0 \\ v_y(x,0)=0 \\ v(x,x)=0\end{cases}\qquad * v,\ \int_D \dots$$

1pt

$$\Rightarrow\quad v v_{xx}+v v_{yy}=0 \iff (v v_x)_x+(v v_y)_y-v_x^2-v_y^2=0$$

$$\Rightarrow\quad \int_D \nabla\cdot\binom{v v_x}{v v_y}dV-\int_D(v_x^2+v_y^2)\,dV=0$$

$$\Rightarrow\quad \int_{\partial D}\binom{v v_x}{v v_y}\cdot\underline{n}\,dV-\int_D(v_x^2+v_y^2)\,dV=0$$

1pt

$$\Rightarrow\quad \int_{\partial D}\underbrace{v.\frac{\partial v}{\partial n}}_{\text{on }\partial V:\ v\text{ or }\frac{\partial v}{\partial n}=0}dV-\int_D v_x^2+v_y^2\,dV=0$$

$$\Rightarrow\quad \int_D v_x^2+v_y^2\,dV=0\quad\Rightarrow\quad v(x,y)=\text{const}$$

$$\underset{v(x,x)=0}{\Rightarrow}\ \text{const}=0\ \Rightarrow\ v\equiv 0\ \Rightarrow\ u_1\equiv u_2\ \Rightarrow\ \dots\ \Rightarrow$$

solution is unique

1pt

Exercise 4 :

a) traffic flow, u car density, f initial car density .. 1pt

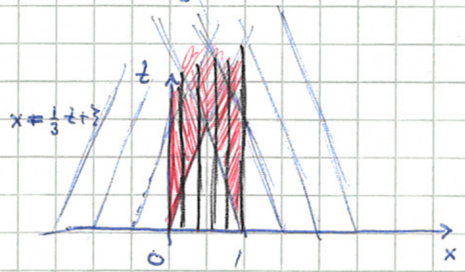
b)

$$\frac{dt}{ds} = 1 \qquad \text{with for } s=0 : \quad t = 0 \qquad\qquad t = s$$

$$\frac{dx}{ds} = 1-2u \qquad\qquad x = \xi \quad \Rightarrow \quad x = (1-2f_0)s + \xi$$

$$\frac{du}{ds} = 0 \qquad u = \begin{cases} \frac{1}{3} & \text{for } \xi<0 \\ \frac{1}{2} & \text{for } 0<\xi<1 \\ \frac{2}{3} & \text{for } \xi>1 \end{cases} \qquad u = f_0 = \begin{cases} \frac{1}{3} & \text{for } \xi<0 \\ \frac{1}{2} & \text{for } 0<\xi<1 \\ \frac{2}{3} & \text{for } \xi>1 \end{cases}$$

1pt



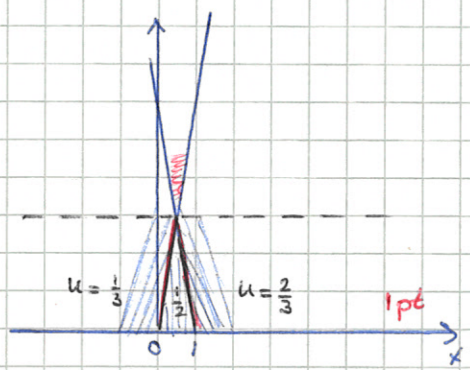
//// intersecting char. ⇒ shock waves

$$\frac{dx_s}{dt} = \frac{[q]}{[e]} \quad \text{with } x_{s,1}(0) = 0, \text{ and } x_{s,2}(0) = 1$$

$$\frac{dx_{s,1}}{dt} = \frac{[u-u^2]}{[u]} = \frac{\frac{1}{2}-\frac{1}{4}-\left(\frac{1}{3}-\frac{1}{9}\right)}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} \Rightarrow x_{s,1}(t) = \frac{1}{6}t + c_1 \underset{x_{s,1}(0)=0}{\Rightarrow} x_{s,1}(t) = \frac{1}{6}t$$

$$\frac{dx_{s,2}}{dt} = \frac{[u-u^2]}{[u]} = \frac{\frac{2}{3}-\frac{4}{9}-\left(\frac{1}{2}-\frac{1}{4}\right)}{\frac{2}{3}-\frac{1}{2}} = \frac{-\frac{1}{36}}{\frac{1}{6}} = -\frac{1}{6} \Rightarrow x_{s,2}(t) = -\frac{1}{6}t + c_2 \underset{x_{s,2}(0)=1}{\Rightarrow} x_{s,2}(t) = -\frac{1}{6}t + 1$$

1pt



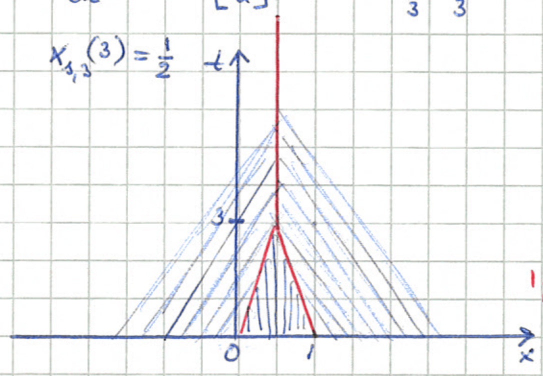
1pt

intersecting shockwaves when $-\frac{1}{6}t + 1 = +\frac{1}{6}t \Rightarrow t = 3$ 1pt

at $t=3$ we have a new shock wave (two densities $\frac{1}{3}$ and $\frac{2}{3}$)

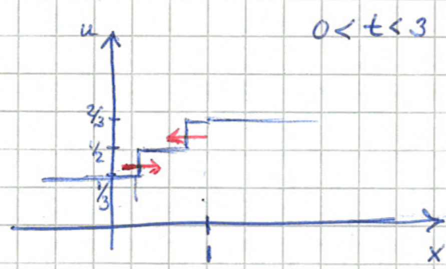
$$\frac{dx_{s,3}}{dt} = \frac{[u-u^2]}{[u]} = \frac{\frac{2}{3}-\frac{4}{9}-\left(\frac{1}{3}-\frac{1}{9}\right)}{\frac{2}{3}-\frac{1}{3}} = \frac{0}{\frac{1}{3}} \Rightarrow x_{s,3}(t) = c_3 \Rightarrow x_{s,3}(t) = \frac{1}{2}$$

$x_{s,3}(3) = \frac{1}{2}$

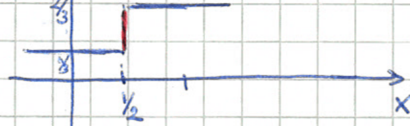


1pt

$0<t<3$



$t \geq 3$



1pt

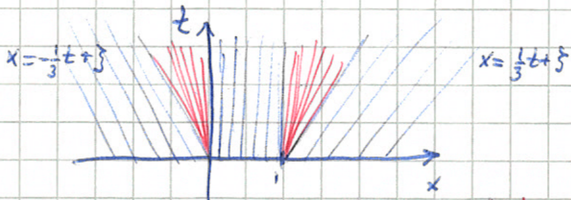
c)

$$\frac{dt}{ds} = .. \qquad\qquad t = s$$

$$\frac{dx}{ds} = .. \quad \Rightarrow \quad x = (1-2f_0)s + \xi$$

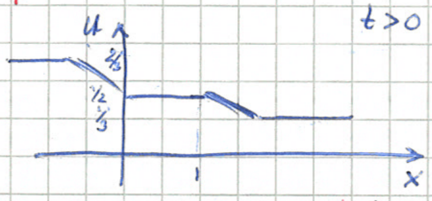
$$\frac{du}{ds} = .. \qquad u = f_0 = \begin{cases} \frac{2}{3} & \text{for } \xi<0 \\ \frac{1}{2} & \text{for } 0<\xi<1 \\ \frac{1}{3} & \text{for } \xi>1 \end{cases}$$

1pt



expansion waves. 1pt

$t>0$



or formulas

1pt