Exam Partial Differential Equations TW2070
Tuesday, June 25, 2019, 13:30 – 16:30 h

Key points of attention:

1. On each sheet of paper that you hand in, you should clearly mention your name and student registration number.
2. It is allowed to use the Laplace- and Fourier transform tables. These tables are added to this exam.
3. This exam consists of 4 exercises. For each exercise 10 points in total can be obtained as indicated in the exercise. Exam grading = number of obtained points divided by 4.

Exercise 1:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} + Q(x,t) \quad , && 0 < x < L, t > 0, \\ &u_x(0,t) = 0\,, && t \geq 0, \\ &u_x(L,t) = -Au(L,t), && t \geq 0, \\ &u(x,0) = f(x), && 0 < x < L, \end{aligned}$$

where $Q(x,t)$ and $f(x)$ are sufficiently smooth functions, and where L , and $A \geq 0$ are constants.

1pt a) Give a physical interpretation of the initial-boundary value problem.

2pt b) Assume that $Q(x,t) \equiv 1$. Determine the equilibrium solution (that is determine u for $t \to \infty$)for those values of A for which an equilibrium solution exists.

2pt c) Take $A = 0$ and $Q \equiv 0$. Determine $u(x,t)$.

2pt d) Describe how the solution of the initial-boundary value problem can be determined for arbitrary A, and $Q(x,t)$. The solution does not have to be computed, but clearly indicate how the solution can be computed (that is, give a description of the method).

3pt e) Prove that the initial-boundary value problem has at most one solution (that is, give a uniqueness proof).

Exercise 2:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_{tt} = u_{xx}, && x > 0, t > 0, \\ &au_x(0,t) = bu(0,t) + cu_t(0,t) - h(t), && t \geq 0, \\ &u(x,0) = f(x), u_t(x,0) = g(x), && x > 0, \end{aligned}$$

where *h*, *f* and *g* are sufficiently smooth functions, and in which *a*, *b* and *c* are non-negative constants.

1pt a) Give a physical interpretation of the problem.

3pt b) Take $a = 1, b = c = o,\ f \equiv 0,$ and $g \equiv 0.$ The general solution of the partial differential equation is given by: $u(x,t) = F(x-t) + G(x+t).$

Determine the solution of the initial-boundary value problem by using the general solution of the PDE.

3pt c) Take $b=1, a=c=0, h\equiv 0,$ and $g\equiv 0.$ Extend the problem to a problem on $-\infty<x<\infty,$ and determine the function $u(x,t)$ by using the Fourier-transform method.

3pt d) Take $a=1, b=0, c=0, f\equiv 0, g\equiv 0.$ Determine $u(x,t)$ by using the Laplace-transform method.

<u>Exercise 3:</u>

Consider the following boundary value problem for the twice continuously differentiable function $u(x,y)$:

$$\begin{aligned} u_{xx}+u_{yy}&=Q(x,y), && 0<y<x<\infty, \\ u_y(x,0)&=f(x), && 0<x<\infty, \\ u(x,x)&=g(x), && o<x<\infty, \end{aligned}$$

where Q, f, and g are sufficiently smooth functions.

1pt a) Give a physical interpretation of the problem.

3pt b) The Green's function for the Laplace operator in $\mathbb{R}^2$ is given by

$$G(\underline{x};\underline{x}_0)=\frac{1}{2\pi}\ln|\underline{x}-\underline{x}_0|.$$

Determine the Green's function for the given boundary value problem.

3pt c) Determine $u(x,y)$.

3pt d) Prove that the solution of the boundary value problem is unique.

<u>Exercise 4:</u>

Consider the following initial value problem for the continuously differentiable function $u(x,t)$:

$$\begin{aligned} u_t+(1-2u)u_x&=0, && -\infty<x<\infty, t>0, \\ u(x,0)&=f(x), && -\infty<x<\infty, \end{aligned}$$

where $f(x)$ is a given function.

1pt a) Give a (physical) interpretation of the problem.

6pt b) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x)=\begin{cases} \frac{1}{3} & \text{for} \quad x<0, \\ \frac{1}{2} & \text{for} \quad 0<x<1, \\ \frac{2}{3} & \text{for} \quad x>1. \end{cases}$$

3pt c) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x)=\begin{cases} \frac{2}{3} & \text{for} \quad x<0, \\ \frac{1}{2} & \text{for} \quad 0<x<1, \\ \frac{1}{2} & \text{for} \quad x>1. \end{cases}$$