TUDelft

Exam Partial Differential Equations AM 2070
Tuesday, June 23, 2020, 13.30 – 16.30 h

Key points of attention:

1) This exam will be an open book exam. This means that you are only allowed to use your book, lecture notes, and the homework exercises you made.
2) You will have at most 15 minutes after the exam for uploading your exam in one (readable) pdf file in Brightspace/assignment/EXAM AM2070 June 2020. So, your exam has to be handed in before 16.45 h.
3) This exam will be a pass/fail exam. When your work has been graded, a number of students will be randomly chosen to have a remote check of 15 minutes.
4) On the exam you have to write down your name and student ID number, and you have to write down and to undersign the following statement: I declare that I have made this examination on my own, with no assistance, and in accordance with the TU Delft policies on plagiarism, cheating, and fraud.
5) This exam consists of 4 exercises. For each exercise 10 points in total can be obtained as indicated in the exercise. Exam grading = number of points divided by 4. The final grading for this course is in line with the earlier determined rules (see also earlier announcements on Brightspace).

Exercise 1:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} + Q(x,t), && 0 < x < L, t > 0, \\ &Au_x(0,t) = u(0,t), && t \geq 0, \\ &u_x(L,t) = -Bu(L,t), && t \geq 0, \\ &u(x,0) = f(x), && 0 < x < L, \end{aligned}$$

where $Q(x,t)$ and $f(x)$ are sufficiently smooth functions, and where $L > 0$, $A \geq 0$, and $B \geq 0$ are constants.

1pt a) Give a physical interpretation of the initial-boundary value problem.

2pt b) Assume that $Q(x,t) = e^{-2x}$. Determine the equilibrium solution (that is, determine u for $t \to \infty$) for those values of A and B for which an equilibrium solution exists.

2pt c) Take $A = B = 0$, and $Q(x,t) \equiv 0$. Determine $u(x,t)$.

2pt d) Describe how the solution of the initial-boundary value problem can be determined for arbitrary A, B, $f(x)$, and $Q(x,t)$. The solution does not have to be computed, but clearly indicate how the solution can be computed (that is, give a description of the method and the steps that have to be made).

3pt e) Prove that the initial-boundary value problem has at most one solution (that is, give a uniqueness proof).

Exercise 2:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_{tt} = u_{xx}\,, && x>0,\ t>0,\\ &au_x(0,t) = bu(0,t) - h(t), && t \ge 0,\\ &u(x,0) = f(x),\ u_t(x,0) = g(x), && x>0, \end{aligned}$$

where *h, f,* and *g* are sufficiently smooth functions, and in which *a* and *b* are non-negative constants.

1pt a) Give a physical interpretation of the problem.

3pt b) Take $a = b = 1,\ f \equiv 0,$ and $g \equiv 0$. The general solution of the partial differential equation is given by: $u(x,t) = F(x-t) + G(x+t)$.
Determine the solution of the initial-boundary value problem by using the general solution of the PDE.

3pt c) Take $a = 1,\ b = 0,\ h \equiv 0,$ and $f \equiv 0$. First extend the problem to a problem on $-\infty < x < \infty$. Then, determine the function $u(x,t)$ by using the Fourier-transform method.

3pt d) Take $a = 0,\ b = 1,\ f(x) \equiv 0$ and $g \equiv 0$. Determine $u(x,t)$ by using the Laplace-transform method.

Exercise 3:

Consider the following boundary value problem for the twice continuously differentiable function $u(x,y,z)$:

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} = Q(x,y,x)\,, && x>0,\ y>0,\ z>0,\\ &u_z(x,y,0) = 0\,, && x>0, y>0,\\ &u(0,y,z) = f(y,z)\,, && y>0, z>0,\\ &u(x,0,z) = 0\,, && x>0, z>0, \end{aligned}$$

where *Q*, and *f* are sufficiently smooth functions, and where *u, Q,* and *f* (as well as their derivatives) tend to zero for $|\underline{x}| \to \infty$.

1pt a) Give a physical interpretation of the problem.

3pt b) The Green's function for the Laplace operator in $\mathbb{R}^3$ is given by

$$G(\underline{x};\underline{x}_0) = \frac{-1}{4\pi|\underline{x}-\underline{x}_0|}.$$

Determine the Green's function for the given boundary value problem.

3pt c) Determine $u(x,y,z)$.

3pt d) Prove that the solution of the boundary value problem is unique.

<u>Exercise 4:</u>

Consider the following initial value problem for the continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_t + (1-2u)u_x = 0, \quad && -\infty < x < \infty,\ t > 0, \\ &u(x,0) = f(x), && -\infty < x < \infty, \end{aligned}$$

where $f(x)$ is a given function.

1pt a) Give a (physical) interpretation of the problem.

4pt b) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} 1 & \text{for } x < 0, \\ \frac{3}{4} & \text{for } 0 < x < 1, \\ \frac{1}{4} & \text{for } x > 1. \end{cases}$$

5pt c) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x < 0, \\ \frac{3}{4} & \text{for } 0 < x < 1, \\ 1 & \text{for } x > 1. \end{cases}$$