Exam Relaxations and Heuristics (WI4515)

21 January 2020, 13.30–16.30 (3 hours).

The exam consists of 5 questions worth 10 points each. Your grade is given by $1 + \frac{9p}{50}$, where p is the total number of points obtained.
Devices, notes, books etc. are **not** permitted.
The total number of pages of this exam is 2. **Good luck!**

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1. Formulate the following as mixed integer programs:

[3pts] (a) $u = \min\{x_1, x_2\}$, assuming that $0 \le x_1, x_2 \le C$.

[3pts] (b) $v = |x_1 - x_2|$, where $0 \le x_1, x_2 \le C$.

[4pts] (c) the set $X \setminus \{x^*\}$, where $X = \{ x \in \mathbb{Z}^n : Ax \le b \}$ and $x^* \in X$.

2. In each of the examples below a set X and a point (x^*, y^*) are given. Find a valid inequality for X cutting off (x^*, y^*). You should prove that the inequality is indeed valid!

[3pts] (a) $X = \{ (x, y) \in \mathbb{R}^3_+ \times \{0,1\}^3 : x_1 + x_2 + x_3 \le 7,\ x_1 \le 3y_1,\ x_2 \le 5y_2,\ x_3 \le 6y_3 \}$, $(x^*, y^*) = (2, 5, 0, 2/3, 1, 0)$.

[3pts] (b) $X = \{ (x, y) \in \mathbb{R}^3_+ \times \{0,1\}^3 : 7 \le x_1 + x_2 + x_3,\ x_1 \le 3y_1,\ x_2 \le 5y_2,\ x_3 \le 6y_3 \}$, $(x^*, y^*) = (2, 5, 0, 2/3, 1, 0)$.

[4pts] (c) $X = \{ (x, y) \in \mathbb{R}^6_+ \times \{0,1\}^6 : x_1 + x_2 + x_3 \le 4 + x_4 + x_5 + x_6,\ x_1 \le 3y_1,\ x_2 \le 3y_2,\ x_3 \le 6y_3,\ x_4 \le 3y_4,\ x_5 \le 5y_5,\ x_6 \le y_6 \}$, $(x^*, y^*) = (3, 3, 0, 0, 2, 0; 1, 1, 0, 0, 2/5, 0)$.

3. A *stable set* in a graph $G = (V, E)$ is a subset of V no two vertices in which are adjacent. A *clique* in G is a subset of V in which any two vertices are adjacent. The *stable set polytope* of G is the convex hull of the incidence vectors of stable sets in G.

   An *odd hole* is a cycle in G with an odd number of vertices and no edges between nonadjacent vertices of the cycle.

[5pts] (a) Show that if H is the vertex set of an odd hole, then

$$x(H) = \sum_{v \in H} x_v \le (|H| - 1)/2$$

is a valid inequality for the stable set polytope.

[5pts] (b) Given a graph $G = (V, E)$, consider the polytopes

$$P = \{ x \in \mathbb{R}^E_+ : x(C) \le 1 \text{ for every clique } C \}$$

and

$$Q = \{ x \in P : x(H) \le (|H| - 1)/2 \text{ for every vertex set } H \text{ of an odd hole} \}.$$

Obviously, the stable set polytope is a subset of Q, which is a subset of P. Show a graph G for which P is a strict superset of Q, that is, $Q \subsetneq P$.

CONTINUE ON THE OTHER SIDE...

[10pts] 4. Suppose we are given demands d_t^k for items $k = 1, \dots, K$ over a time horizon $t = 1, \dots, T$. All items must be produced on a single machine; the machine can produce only one item in each period and has a capacity C_t^k if item k is produced in period t. Given production, storage, and set-up costs for each item in each period, we wish to find a minimum cost production plan. This problem can be formulated as:

$$\begin{aligned} \min \textstyle\sum_{k=1}^{K} \sum_{t=1}^{T} \left(p_t^k x_t^k + h_t^k s_t^k + f_t^k y_t^k\right) & \\ \sum_{k=1}^{K} y_t^k &\leq 1 \text{ for } t = 1, \dots, n, \\ s_{t-1}^k + x_t^k &= d_t^k + s_t^k \text{ for } t = 1, \dots, n, k = 1, \dots, K, \\ x_t^k &\leq C_t^k y_t^k \text{ for } t = 1, \dots, n, k = 1, \dots, K, \\ \left(x^k, s^k, y^k\right) &\in R_+^n \times R_+^n \times B^n. \end{aligned}$$

Formulate the integer programming master and subproblems for this problem.

[10pts] 5. Consider the $0-1$ knapsack problem: $z = \max \sum_{j \in N} c_j x_j : \sum_{j \in N} a_j x_j \leq b, x \in B^n$ with $a_j > 0$ for $j \in N$. Consider a greedy heuristic that chooses the best solution between 1) rounding down the linear programming solution and 2) the best solution in which just one variable is set to one. Show that $z^G \geq \frac{1}{2} z$.

End of the exam