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Exam Statistical Inference (WI4455)
April 17, 2019, 13.30–16.30

- This is an open book exam.
- Unless stated differently, always add an explanation to your answer.
- Please write on top of your exam:

  I declare that I have made this examination on my own, with no assistance and in accordance with the TU Delft policies on plagiarism, cheating and fraud.

  and add your signature below.
- You can write down your answers on your own piece of paper, but please write down your name, student number, and course number on the first page and a page number on each piece of paper. When you are done, take a photo of your work or scan your work and send me your work as one pdf-file by email f.h.vandermeulen@tudelft.nl. Please ensure that a photo of your student ID card is also in the file (on one of the photos that are combined to a pdf file).
- In case of questions about the exam, or technical problems at an earlier stage, send me an email at f.h.vandermeulen@tudelft.nl; I'll be monitoring my inbox the entire duration of the exam.

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1. Define the probability density function

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x} \mathbf{1}_{[0,\infty)}(x),$$

where α and β are strictly positive parameters and Γ is the Gamma-function.

   (a) Verify that f is of exponential family type.

   (b) Suppose $X_1,\ldots,X_n$ are independent and identically distributed with density f. Derive a sufficient statistic for (α,β) and argue why it is complete. *Hint: natural parameter space.*

   (c) We now consider the Bayesian viewpoint. Suppose $X_1,\ldots,X_n$ are independent with density f, conditional on β. Assume $\beta \sim \mathit{Exp}(2)$. Derive an expression for the posterior density of β, while assuming α is fixed (that is, known).

   (d) Now suppose also α is endowed with a prior distribution: assume $\alpha \sim \mathit{Exp}(1)$. Give the steps of an MCMC-algorithm to draw from the posterior of (α,β).

2. (a) Suppose $X \sim N(0,\psi)$. Derive an expression for the Fisher-information $I(\psi)$. Note that $\psi > 0$ is the variance of the Normal distribution, so there is no square appearing.

(b) Suppose $h : (0, \infty) \to (0, \infty)$ is bijective, differentiable, with differentiable inverse. Show that if we parametrise by $h(\psi)$ instead of ψ, then the Fisher information satisfies $I_h(\psi) = h'(\psi)^2 I(h(\psi))$. Here $I_h(\psi)$ denotes the Fisher-information when $h(\psi)$ is used as parametrisation.

3. Assume p pairs of observations $(X_1, Y_1), \ldots, (X_p, Y_p)$, where all pairs are assumed conditionally independent upon parameters $\theta_1, \ldots, \theta_p$. We further assume $X_i \sim N(0, 1)$ and $Y_i \mid X_i = x \sim N(\theta_i x, 1)$.

   (a) Following a Bayesian approach, assume that the parameters are random quantities themselves. Hence, write the parameters as $\Theta_1, \ldots, \Theta_p$ and assume these random variables are independent with $N(0, \tau^2)$-distribution. Find The Bayes estimator for Θ_i under squared error loss.

   (b) Determine $\operatorname{E} Y_i^2$ and use this result to define a method of moments estimator for τ^2.

   (c) Derive empirical Bayes estimators for θ_i ($i \in \{1, \ldots, p\}$) by combining parts (a) and (b).

   (d) Now consider a frequentist approach and derive the maximum likelihood estimator for θ_i ($i \in \{1, \ldots, p\}$)

4. Suppose $X_1, \ldots, X_n$ are identically distributed and independent, conditional on the parameter Θ. Assume we endow Θ with a prior distribution.

   (a) For $c > 0$ consider the loss function

   $$L_c(\theta, a) = \Psi(c(\theta - a)) \quad \text{with} \quad \Psi(x) = e^x - x - 1.$$

   We consider the Bayes rule for estimating θ using L_c. What is considered more costly, under- or over estimation of θ?

   (b) Show that the Bayes rule satisfies

   $$d_c(x_1, \ldots, x_n) = -\frac{1}{c} \log \int e^{-c\theta} f_{\Theta \mid X_1, \ldots, X_n}(\theta_1 \mid x_1, \ldots, x_n) \mathrm{d}\,\theta.$$

   (c) What is the Bayes rule in the limit where we let $c \downarrow 0$? Sketch the main argument, you don't have to be fully rigorous.