Exam Statistical Inference (WI4455)
January 23, 2020, 9.00–12.00

Using books or notes is not allowed at the exam.
Unless stated diferently, always add an explanation to your answer.

1. Let g be a density function supported on $[0, 1]$ (i.e. its density is zero outside $[0, 1]$). Denote $\mu = \int x g(x) \mathrm{d}\, x$ and $\sigma^2 = \int (x - \mu)^2 g(x) \mathrm{d}\, x$. *In this exercise we consider g fixed and known.*

   Let $\theta \in [0, 1]$ and define the probability density

$$f(x \mid \theta) = \begin{cases} \theta + (1 - \theta) g(x) & \text{if } x \in [0, 1] \\ 0 & \text{if } x \notin [0, 1] \end{cases}.$$

   Assume $X_1, \ldots, X_n$ are independent, each with density f.

   (a) [1 pt]. Give an expression for the score-function (this is the derivative of the log-likelihood).

   (b) [1 pt]. Show that $\mathrm{E}_\theta X_1 = \Psi_\mu(\theta)$, with $\Psi_\mu(\theta) = \mu + (1 - \mu)\theta$.

   (c) [2 pt]. Define the estimator $\hat{\Theta}_n$ by the relation $\bar{X}_n = \Psi_\mu(\hat{\Theta}_n)$. Derive an expression for $\hat{\Theta}_n$ and show it is unbiased for θ.

   (d) [3 pt]. Using the central limit theorem, give the limiting distribution of $\sqrt{n}(\hat{\Theta}_n - \theta)$ under $\mathbb{P}_\theta$ and show that if $\mu < 1$

$$\lim_{n \to \infty} \mathbb{P}_\theta(\hat{\Theta}_n < 0) = 0.$$

   (e) [2 pt]. In the remainder of the exercise we assume $n = 1$, so just one observation X_1. Show that the maximum likelihood estimator is given by

$$\hat{\Theta}_{\text{MLE}} = \begin{cases} 0 & \text{if } \ g(X_1) > 1 \\ 1 & \text{if } \ g(X_1) < 1 \end{cases}$$

   ? $g(x) = 0$ if $x \notin [0,1]$

   (f) [2 pt]. We shift perspective to the Bayesian view. Hence we consider $X_1 \mid \Theta = \theta \sim f(x \mid \theta)$ and employ a prior on the parameter θ that is supported on $[0,1]$ with density denoted by f_Θ. Show that the posterior density satisfies

$$f_{\Theta|X_1}(\theta \mid x) = \frac{f(x \mid \theta) f_\Theta(\theta)}{\pi_1 + (1-\pi_1) g(x)} \mathbf{1}_{[0,1]}(\theta).$$

where π_1 is the prior mean.

(g) [2 pt]. Suppose the prior on Θ is taken to be the uniform distribution on $[0,1]$. Express the posterior mean in terms of $g(X_1)$.

2. Let $\theta \in (0,\infty)$ be an unknown parameter and X be a random variable such that $\mathrm{E}_\theta X = \theta$ and $\mathrm{var}_\theta X = \nu(\theta)$, where $\nu(\theta)$ is known and specified below. Consider estimation of θ by a decision rule within the class $\mathcal{D}$ defined by

$$\mathcal{D} = \{d_a(X) = aX,\ a \in (0,1]\}.$$

Assume squared error loss, that is, $L(\theta, d_a) = (\theta - aX)^2$.

(a) [3 pt]. For $\nu(\theta) = \theta^2$, calculate the risk function of d_a, and show that there is a value of a which is optimal, no matter the value of θ.

(b) [1 pt]. Show that d_1 is inadmissible (for the given loss-function). *Hint: consider also* $d_{1/2}$.

(c) [3 pt]. Suppose $\nu(\theta) = \theta^k$ where k is a positive integer. Show that the Bayes risk of the decision rule d_a is given by $a^2\, k! + 2(a-1)^2$, when Θ has prior density $f_\Theta(\theta) = e^{-\theta}\mathbf{1}_{[0,\infty)}(\theta)$. In addition, compute the Bayes decision rule.

You can use the fact that $\int_0^\infty x^n e^{-x} \mathrm{d}\, x = n!$ *for positive integers* n.

(d) [1 pt]. Suppose again that $\nu(\theta) = \theta^2$. Are the minimax rule and Bayes rule (that you derived in part (c)) the same?

(e) [1.5 pt]. Show that the Bayes rule does not depend on the chosen prior distribution on Θ if $\nu(\theta) = \theta^2$.

3. Consider the following hierarchical model

$$\begin{aligned} X_1, \ldots, X_n \mid \Theta = \theta &\overset{\text{ind}}{\sim} Pois\,(\theta) \\ \Theta &\sim Ga\,(\alpha, \beta)\,, \end{aligned}$$

where $Ga\,(\alpha, \beta)$ denotes the Gamma-distribution with parameters α and β. That is,

$$f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \mathbf{1}_{[0,\infty)}(\theta),$$

where Γ denotes the Gamma function. Recall that under the specified model $\mathrm{P}_\theta(X_i = x) = e^{-\theta} \frac{\theta^x}{x!}$, when $x \in \{0, 1, \ldots\}$.

(a) [3 pt]. Show that the posterior distribution of Θ is a Gamma distribution. Specify its parameters.

(b) [2 pt]. Show that the marginal density of $X = (X_1, \ldots, X_n)$ equals

$$f_X(x) = \frac{b^a \Gamma(a+s)}{\Gamma(a) \prod_{i=1}^n (x_i!)(b+n)^{a+s}},$$ (α, β)

where $s = \sum_{i=1}^n x_i$.

(c) [3 pt]. Assume $\alpha = 2$ and that we further endow β with a prior distribution with density $p(\beta) = e^{-\beta} \mathbf{1}_{[0,\infty)}(\beta)$. Give the steps of the Gibbs sampler for sampling from the posterior distribution of (θ, β).

4. Suppose $X \sim Pois\,(\theta)$.

(a) Verify that $\varphi(X)$ is an unbiased estimator for $e^{-3\theta}$ if

$$\sum_{k=0}^{\infty} \varphi(k) \frac{\theta^k}{k!} = e^{-2\theta}.$$

(b) Prove that $(-2)^X$ is UMVU for θ. *Hint: You may use the trivial fact that X is a complete and sufficient statistic for θ.*