Solution of Exercise 1

(a) $\tilde{x} = (25\ \ 25\ \ 2)^\top$ and $\tilde{u} = 4$.

(b) $\dot{z} = Az + Bv,\ w = Cz + Dv$ with

$$A = \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & -\frac{1}{5} & 5 \\ 0 & 0 & -2 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ D = 0.$$

Solution of Exercise 2

(a) The transfer function of the interconnection equals $\dfrac{s^2+s-2}{s^3+3s^2+(2-k)s+k}$.
A state space description of Σ is given by $\dot{x} = Ax + Bv, z = Cx$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & k-2 & -3 \end{pmatrix},\quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad C = \begin{pmatrix} -2 & 1 & 1 \end{pmatrix}.$$

(c) Using the Routh criterion it follows that for $k \in (0, 1\frac{1}{2})$ the transfer function is stable.

Solution of Exercise 3

(a) $R = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \end{pmatrix}$, rank $R = 2$, im $R =$ span $\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$,
independent of α, β.

(b) $W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}$, rank $W = 2$, ker $W =$ span $\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$,
independent of α, β.

(c) Transformation matrix $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ yields $\tilde{A} = T^{-1}AT =$

$$\left(\begin{array}{cc|cc} 0 & \alpha & 1 & -1 \\ 1 & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & -\beta \\ 0 & 0 & -1 & 0 \end{array}\right),\ \tilde{B} = T^{-1}B = \left(\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right) \text{ and } \tilde{C} = CT = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \end{array}\right).$$

(d) From the structure of the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ it follows that $Ce^{At}B = \tilde{C}e^{\tilde{A}t}\tilde{B} = 0$. This also follows from (a) and (b) as the controllable (=reachable) subspace is contained in, actually equals, the unobservable subspace.

(e) If $\alpha = \beta = 0$, all eigenvalues of A and $\tilde{A}$ are located at 0, look at $\tilde{A}$. Hence, the algebraic multiplicity of the eigenvalue 0 is 4. However, also from $\tilde{A}$, it follows that the geometric multiplicity of the eigenvalue 0 is at most 3. Hence, the system is unstable.

Solution of Exercise 4 Note that A has eigenvalues $-2, -1$ and 1.

(a) For $\lambda = 1$, rank $(A - \lambda I, B) = 3$ and rank $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = 3$. Verify this! (Why does it suffice to look at $\lambda = 1$ only?) It follows that the system is stabilizable and detectable.

(b) With $F = \begin{pmatrix} 0 & -9 & -30 \end{pmatrix}$ the eigenvalues of $A + BF$ are located at $-2, -4, -5$. Note that -2 is a fixed eigenvalue. The others can be placed arbitrarily.

(c) Simple inspection show that $K = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ is such that $A - KC$ has eigenvalues at the requested locations.

(d) For the dynamic controller, see (5.10) in the course notes. The eigenvalues of the closed loop system are located at $-2, -4, -5, -1, -2, -3$. See also the text below equation (5.12) in the course notes.

Solution of Exercise 5

(a) The statement is false. Take $A = -1$ and $B = 0$, then (A, B) is stabilizable, but (A^2, B) is not.

(b) The statement is true. For instance, take $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $y(t) = \sin 2t$ for any input function, hence also for $u(t) = e^{-t}$.

(c) The statement is false. Consider the transfer functions $\frac{(s-1)}{(s+1)(s+2)}$ and $\frac{1}{(s-1)}$. The first one is stable, the second one is unstable. Then the series connection has the transfer function $\frac{1}{(s+1)(s+2)}$, which is stable.

(d) The statement is true. In fact, the statement follows from the fact that $\det(sI - A) = \det(sI - A)^\top = \det(sI - A^\top)$. For the characteristic polynomial of a matrix in companion form see Exercise 3.5.14 in the course notes.